Topological state sum models in four dimensions, half-twists and their applications

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Abstract

Various mathematical tools are developed with the aim of application in mathematical physics.

In the first part, a new state sum model for four-manifolds is introduced which generalises the Crane-Yetter model. It is parametrised by a pivotal functor from a spherical fusion category into a ribbon fusion category. The special case of the Crane-Yetter model for an arbitrary ribbon fusion category $\mathcal{C}$ arises when we consider the canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C})$ into the Drinfeld centre as the pivotal functor. The model is defined in terms of handle decompositions of manifolds and thus enjoys a succinct and intuitive graphical calculus, through which concrete calculations become very easy. It gives a chain-mail procedure for the Crane-Yetter model even in the case of a nonmodular category. The nonmodular Crane-Yetter model is then shown to be nontrivial: It depends at least on the fundamental group of the manifold. Relations to the Walker-Wang model and recent calculations of ground state degeneracies are established.

The second part develops the theory of involutive monoidal categories and half-twists (which are related to braided and balanced structures) further. Several gaps in the literature are closed and some missing infrastructure is developed. The main novel contribution are "half-ribbon" categories, which combine duals – represented by rotations in the plane by π – with half-twists, which are represented by turns of ribbons by π around the vertical axis. Many examples are given, and a general construction of a half-ribbon category is presented, resulting in so-called half-twisted categories.

Contents

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Part I

Introduction

1 About this thesis

Many branches of mathematics are inspired by physics. Especially geometry draws heavy influences from the physical sciences, and quite a few areas (like Riemannian geometry, gauge theory and symplectic geometry) felt significant advances through research aimed at physical applications. This is the spirit this thesis is written in: Pure mathematics, but with applications to physics and geometry in mind. One of the greatest problems of contemporary fundamental physics is quantum gravity, and also one of the greatest inspirations for the selection of topics; in the hope that the results contained in this thesis may be helpful in the search for an answer to this question.

The other inspiration for this thesis is category theory, to be more specific, the theory of monoidal categories, higher categories and graphical calculus. When higher-dimensional algebra is too hard to carry out, it is often possible to represent morphisms as diagrams and equations as transformations, and to use one's graphical intuition to arrive at a simple proof. This way of working playfully, yet rigorously, is made possible by a deep connection between higher category theory and topology; a connection which continues to inspire many areas, such as topological quantum field theories.

This is probably all that can be said in terms of unifying principles about this thesis. The next two parts are very different:

The second part describes a new state sum model together with a handy graphical calculus. The Crane-Yetter model, originally developed for the study of quantum gravity, arises as a special case. The graphical calculus allows for very succinct and yet precise calculations that can be used to answer open questions and gain new insights about the Crane-Yetter model with ease.

The third part takes diagrams even more serious and develops the field of half-twists further. It may appear as an exercise in pure category theory, but in fact two motivations to study half-twists are Noncommutative Geometry (NcG) – an approach aiming at a unification of geometry, gravity theories and quantum field theory – and two-dimensional state sum models, which seem to be intimately related to NcG.

Since the parts are so different, they deserve an introduction each. Only common preliminaries are explained in the next section.

2 Preliminaries

2.1 Monoidal categories with additional structure

In mathematical physics, one encounters a multitude of linear monoidal categories with additional structure and functors preserving this structure. Usually, the category Vect of finite dimensional vector spaces over $\mathbb{C}$ serves as a trivial example for these. The additional structures often arise as special cases of higher categorical structures, for example, monoidal categories are bicategories with one object and braided categories are in some sense tricategories with one 1-morphism. This beautiful motivation is explained more closely in the literature, e.g. [SP11, section B.3]. Here the definitions are given in a closely related manner by discussing their suitability for graphical calculus. Monoidal categories are needed for a graphical calculus of one-dimensional ribbon tangles in two dimensions; similarly one needs the braided structure for evaluating tangle diagrams in three dimensions. An overview of most commonly used definitions of monoidal categories with additional structure, together with their graphical calculus, can be found in [Sel10].

Semisimple and linear categories

Definition 2.1. A **$\mathbb{C}$-linear category** is a category enriched in $\mathrm{Vect}_{\mathbb{C}}$. If not mentioned otherwise, all categories in this work are $\mathbb{C}$-linear categories and all functors are **linear functors**, that is, functors in the enriched category. This implies that they are linear on the morphism spaces and preserve direct sums.

Definition 2.2. An object $X \in \mathrm{ob}\,\mathcal{C}$ is called **simple** if $\mathcal{C}(X,X) \cong \mathbb{C}$.

Examples 2.3.

- In Vect, $\mathbb{C}$ is the only simple object up to isomorphism.
- In $\mathrm{Rep}(G)$, the representation category of a finite group G, the simple objects are the irreducible representations.

Note that simple objects are called scalar objects in [Pet08].

Definition 2.4. A linear category $\mathcal{C}$ is called **semisimple** if it has biproducts, idempotents split (i.e. it has subobjects) and there is a set of inequivalent simple objects $\Lambda_{\mathcal{C}}$ such that for each pair of objects X, Y, the map

$$\Phi\colon \bigoplus_{Z\in\Lambda_{\mathcal{C}}} \mathcal{C}(X,Z) \otimes \mathcal{C}(Z,Y) \to \mathcal{C}(X,Y)$$

obtained by composition and addition is an isomorphism. If the set $\Lambda_{\mathcal{C}}$ is finite, then the category is called **finitely semisimple**.

Remark 2.5. The requirements of biproducts and subobjects in this definition are not very restrictive. According to the discussion in [Mü03a], any category that satisfies all of the conditions in the definition of a semisimple category except for the existence of biproducts and subobjects can be embedded as a full subcategory of a semisimple category.

Example 2.6. For every finite group G, $\mathrm{Rep}(G)$ is finitely semisimple. The simple objects are the irreducible representations.

Lemma 2.7. Let Z_1 and Z_2 be two nonisomorphic simple objects. Then there are no nontrivial morphisms between them, i.e. $\mathcal{C}(Z_1, Z_2) = 0$.

Proof. Decompose $\mathcal{C}(Z_1, Z_2)$ according to Definition 2.4. Both $\mathcal{C}(Z_1, Z_2) \otimes \mathcal{C}(Z_2, Z_2)$ and $\mathcal{C}(Z_1, Z_1) \otimes \mathcal{C}(Z_1, Z_2)$ occur as summands. But since $\mathcal{C}(Z_1, Z_1) \cong \mathcal{C}(Z_2, Z_2) \cong \mathbb{C}$, $\mathcal{C}(Z_1, Z_2) \otimes \mathbb{C}^2$ is a subspace of $\mathcal{C}(Z_1, Z_2)$, which implies that $\mathcal{C}(Z_1, Z_2) \cong 0$. □

Definition 2.8. For a simple object Z and any object X in a linear category, there is a bilinear pairing:

$$
\begin{aligned}
&(-,-)\colon \mathcal{C}(Z, X) \times \mathcal{C}(X, Z) \to \mathbb{C} \\
&(f, g) \cdot 1_Z = g \circ f
\end{aligned}
$$

The $-$ are placeholders.

Lemma 2.9. In a semisimple category, the bilinear pairing is non-degenerate.

Proof. Let $g\colon X \to Z$ such that all $f\colon Z \to X$ satisfy $gf = 0$. Then decompose $1_X = \sum_{Z',i} \alpha^i_{Z'} \alpha_{Z',i}$ according to Definition 2.4, which implies $g = g1_X = g \sum_{Z',i} \alpha^i_{Z'} \alpha_{Z',i}$. From the previous lemma we know that if Z and Z' are not isomorphic then $g\alpha^i_{Z'} = 0$, therefore the sum reduces to $g \sum_i \alpha^i_Z \alpha_{Z,i}$. But $\alpha^i_Z\colon Z \to X$, so by assumption $g\alpha^i_Z = 0$ and therefore $g = 0$.

An analogous argument holds for f. □

Monoidal categories, functors and natural transformations

Definition 2.10. A **monoidal category** consists of:

- A category $\mathcal{C}$,

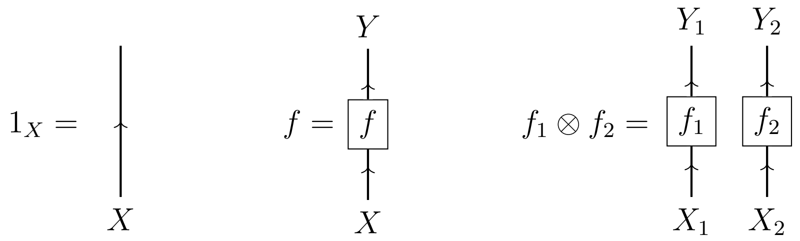
- a functor $-\otimes -\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ called the **monoidal product**,
- a unit object I called the **monoidal identity**,
- natural associativity isomorphisms $\alpha_{X,Y,Z}\colon (X\otimes Y)\otimes Z\to X\otimes(Y\otimes Z)$ and natural unit isomorphisms $\lambda_X\colon I\otimes X\to X$ and $\rho_X\colon X\otimes I\to X$ subject to coherence conditions which can be found e.g. in [Sel10, Section 3.1].

In a **strict monoidal category**, the coherence morphisms α, λ and ρ are all identity morphisms.

If a monoidal category is also linear, $\otimes$ is assumed to be bilinear:

$$(f+g)\otimes h = (-\otimes -)(f+g,h) = f\otimes h + g\otimes h$$
$$f\otimes(g+h) = (-\otimes -)(f,g+h) = f\otimes g + f\otimes h \tag{2.1.1}$$

In the graphical calculus for monoidal categories, morphisms $f\colon X\to Y$ are drawn as boxes and lines in the plane, from the bottom to the top:



(2.1.2)

The upward-pointing arrow on the lines is optional at this point but will be a useful device when duals are introduced. The coherence morphisms are not shown in the diagrammatic calculus. This is due to MacLane's famous coherence theorem which states that any composition of coherence morphisms between two given objects is unique [ML63]. Hence there is no ambiguity in the way the coherence morphisms are inserted. Also, the coherence theorem shows that every monoidal category is monoidally equivalent to a strict monoidal category. Hence one can alternatively view the diagrammatic calculus as determining morphisms in the equivalent strict category. Throughout the paper, monoidal categories (possibly with extra structure) will be indicated by the name of the mere category whenever standard notation for all the additional data is used.

Definition 2.11. A **monoidal functor** is a tuple (F,F^2,F^0), where

- $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between monoidal categories,
- $F^2_{X,Y}\colon FX \otimes_{\mathcal{D}} FY \Rightarrow F(X \otimes_{\mathcal{C}} Y)$ is a natural isomorphism,
- $F^0\colon I_{\mathcal{D}} \to FI_{\mathcal{C}}$ is an isomorphism in $\mathcal{D}$.

F^2 and F^0 are required to commute with the coherence morphisms, see e.g. [Sel10, Section 3.1]. A **monoidal natural transformation** is a natural transformation that commutes with F^0 and F^2.

Note that here F^2 and F^0 are assumed to be isomorphisms. Such functors are also sometimes called "strong monoidal".

Rigid and fusion categories

Definition 2.12. A **duality** is a quadruple $(X, Y, \mathrm{ev}\colon X \otimes Y \to I, \mathrm{coev}\colon I \to Y \otimes X)$ satisfying the "snake identities":

$$(\mathrm{ev} \otimes 1_X) \circ (1_X \otimes \mathrm{coev}) = 1_X$$
$$(1_Y \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes 1_Y) = 1_Y \tag{2.1.3}$$

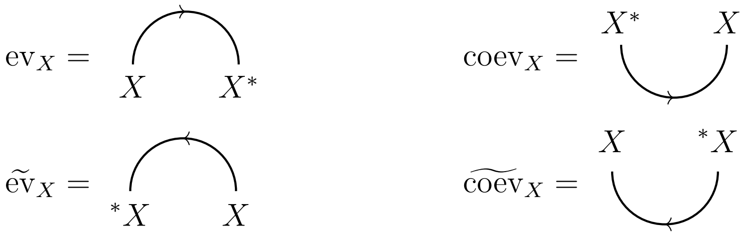
In this situation, $(X, \mathrm{ev}, \mathrm{coev})$ is called the left dual of Y, and $(Y, \mathrm{ev}, \mathrm{coev})$ the right dual of X. The morphisms ev and coev are called "evaluation" and "coevaluation", respectively. (In the context of adjunctions, they are also called "unit" and "counit".)

Definition 2.13. A monoidal category with left (right) duals for every object is called a **left (right) rigid category**. A **rigid**, or "autonomous" category is a category that is left rigid and right rigid, i.e., every object has a left and a right dual.

Definition 2.14. Finitely semisimple rigid categories with simple I are known as **fusion categories**.

In this work, each object X in a rigid category will have a particular choice of duals. The right dual is denoted $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ and the left dual $({}^*X, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$. Pre- and postcomposing morphisms with ev and coev (resp. $\widetilde{\mathrm{ev}}$ and $\widetilde{\mathrm{coev}}$) defines right (resp. left) dual contravariant op-monoidal functors $-^*$ (resp. ${}^*-$). They are contravariant in the sense that source and target are switched, and op-monoidal in the sense that the monoidal product is reversed via canonical isomorphisms $\delta_{X,Y}\colon (X \otimes Y)^* \cong Y^* \otimes X^*$.

Evaluation and coevaluation morphisms are drawn as caps and cups. The arrow in the diagram is an orientation for the line that points to the right for the right duals.



(2.1.4)

The arrow notation means that it is possible to regard the object X as a label on the whole line (rather than one end of it). The convention at the ends of the line is that an upward-pointing arrow indicates X and a downward-pointing arrow X^*.

In this graphical calculus, the snake identities now become:

(2.1.5)

Indeed, every identity of strings that is true as an isotopy in the plane is true for morphisms in a rigid category.

Applying a monoidal functor (F, F^2, F^0) to the snake identities shows that dualities are preserved, i.e. that the following morphism is an evaluation:

$$FX \otimes FY \xrightarrow{F^2_{X,Y}} F(X \otimes Y) \xrightarrow{F\,\mathrm{ev}} FI_{\mathcal{C}} \xrightarrow{(F^0)^{-1}} I_{\mathcal{D}}$$

A similar statement holds for the coevaluation. Proving this requires all the naturality axioms of a monoidal functor.

A standard result on dualities is that any two duals of a given object X are canonically isomorphic. Applying this to F shows [Pfe09] that there are canonical isomorphisms for the right duals

$$u_X \colon F(X^*) \to (FX)^* \tag{2.1.6}$$

determined by F. These satisfy the defining equations

$$\mathrm{ev}_{FX} = \left(F^0\right)^{-1} \circ F\,\mathrm{ev}_X \circ F^2_{X,X^*} \circ \left(1 \otimes u_X^{-1}\right) \tag{2.1.7}$$

$$\mathrm{coev}_{FX} = (u_X \otimes 1) \circ \left(F^2_{X^*,X}\right)^{-1} \circ F\,\mathrm{coev}_X \circ F^0 \tag{2.1.8}$$

There are also separate canonical isomorphisms in a similar way for the left duals.

Pivotal and spherical categories

There exist rigid categories in which every left dual is also a right dual, i.e. $X^* \cong {}^*X$. Since there already exist canonical natural isomorphisms $l_X \colon X \xrightarrow{\cong} ({}^*X)^*$ and $\tilde{l}_X \colon X \xrightarrow{\cong} {}^*(X^*)$ in any rigid category, isomorphisms between left and right duals are equivalent to isomorphisms to the double dual, $X \cong X^{**}$. Choosing such an isomorphism naturally and monoidally for each object leads to the following definition.

Definition 2.15. A **pivotal category** is a right rigid category $\mathcal{C}$ (with chosen right duals) together with a monoidal natural isomorphism $i \colon 1_{\mathcal{C}} \to -^{**}$, the **pivotal structure**. They are also called "sovereign" categories.

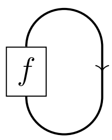
Lemma 2.16. A pivotal category is also left rigid, and thus rigid, with the following choice of left dual:

$$ {}^*X := X^* \tag{2.1.9} $$

$$ \widetilde{\mathrm{ev}}_X := \mathrm{ev}_{X^*} \circ (1_{X^*} \otimes i_X) \tag{2.1.10} $$

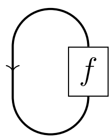
$$ \widetilde{\mathrm{coev}}_X := \left(i_X^{-1} \otimes 1_{X^*}\right) \circ \mathrm{coev}_{X^*} \tag{2.1.11} $$

Definition 2.17. Left and right traces $\mathrm{tr}_L, \mathrm{tr}_R \colon \mathcal{C}(X,X) \to \mathcal{C}(I,I) \cong \mathbb{C}$ can be defined with a pivotal structure:

$\mathrm{tr}_R(f) :=$ 

$$ = \mathrm{ev}_X \circ (f \otimes 1_{X^*}) \circ \widetilde{\mathrm{coev}}_X $$

$$ = \mathrm{ev}_X \circ \left(\left(f \circ i_X^{-1}\right) \otimes 1_{X^*}\right) \circ \mathrm{coev}_{X^*} \tag{2.1.12} $$

$\mathrm{tr}_L(f) :=$ 

$$ = \widetilde{\mathrm{ev}}_X \circ \left(1_{^*X} \otimes f\right) \circ \mathrm{coev}_X $$

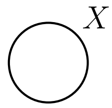
$$ = \mathrm{ev}_{^*X} \circ \left(1_{^*X} \otimes (i_X \circ f)\right) \circ \mathrm{coev}_X \tag{2.1.13} $$

There are pivotal categories for which $\mathrm{tr}_R \neq \mathrm{tr}_L$ for some objects. Spherical categories eliminate this discrepancy.

Definition 2.18. A **spherical category** is a pivotal category with $\mathrm{tr}_R = \mathrm{tr}_L$ for every object. This trace will then simply be called tr. The pivotal structure of a spherical category is also called a "spherical structure". The **dimension** of an object X is defined as $d(X) \coloneqq \mathrm{tr}\,(1_X)$. It is also called "categorical" dimension, or, for representations of Hopf algebras, "quantum" dimension.

The diagram for the dimension of an object is a circle. Note that because of sphericality, it is not necessary to specify a direction on the circle.

$$d(X) = \mathrm{tr}(1_X) = \tag{2.1.14}$$



Note that the dimension of a simple object is known to be nonzero in fusion categories [ENO05]. This follows from the facts that for a simple object Z the spaces $\mathcal{C}(I, Z \otimes Z^*)$ and $\mathcal{C}(Z \otimes Z^*, I)$ have dimension 1, evaluations and coevaluation are non-zero elements of these spaces, and Lemma 2.9.

Remark 2.19. The name "spherical" arises from the fact that the diagram of a morphism can be embedded on the 2-sphere, and every isotopy on the sphere amounts to a relation in the category. The additional axiom of a spherical category corresponds to moving a strand "around the back" of the sphere. However, the spherical axiom implies further identities that don't come from isotopies on the sphere.

Definition 2.20. Let X and Y be two arbitrary objects in a spherical fusion category. The **spherical pairing** of two morphisms $f\colon X \to Y$ and $g\colon Y \to X$ is defined as

$$\langle f, g\rangle \coloneqq \mathrm{tr}(gf) = \mathrm{tr}(fg) \tag{2.1.15}$$

Lemma 2.21. The spherical pairing on a spherical fusion category is nondegenerate.

Proof. With the notation from Definitions 2.4 and 2.20, decompose $f = \sum_{Z,i} \beta^i_Z \alpha_{Z,i}$ and $g = \sum_{Z',j} \delta^j_{Z'} \gamma_{Z',j}$. Then

$$\langle f, g\rangle = \sum_{Z,i,Z',j} \mathrm{tr}\left(\delta^j_{Z'}\gamma_{Z',j}\beta^i_Z\alpha_{Z,i}\right) \qquad = \sum_{Z,i,Z',j} \mathrm{tr}\left(\gamma_{Z',j}\beta^i_Z\alpha_{Z,i}\delta^j_{Z'}\right)$$

But $\gamma_{Z',j}\beta^i_Z$ is a map from Z to Z', and so is non-zero only if $Z = Z'$. In this case, it is equal to $(\beta^i_Z, \gamma_{Z,j})\, 1_Z$, thus the expression reduces to

$$= \sum_{Z,i,j} \left(\beta^i_Z, \gamma_{Z,j}\right) \mathrm{tr}\left(\alpha_{Z,i}\delta^j_Z\right) \qquad = \sum_{Z,i,j} \left(\beta^i_Z, \gamma_{Z,j}\right) \left(\delta^j_Z, \alpha_{Z,i}\right) d(Z)$$

The dimensions $d(Z)$ of simple objects are nonzero, hence with Lemma 2.9 this is non-degenerate. □

Definition 2.22. A **pivotal functor** $F\colon \mathcal{C} \to \mathcal{D}$ is a strong monoidal functor preserving the pivotal structure (and thus the isomorphism between left and right dual) up to canonical isomorphisms. More specifically, the following diagram must commute:

$$\begin{array}{ccc} FX & \xrightarrow{i_{FX}} & (FX)^{**} \\ {\scriptstyle Fi_X}\downarrow & & \downarrow{\scriptstyle u_X^*} \\ F(X^{**}) & \xrightarrow{u_{X^*}} & (F(X^*))^* \end{array} \tag{2.1.16}$$

In this diagram, u is the canonical isomorphism from (2.1.6).

Lemma 2.23. Pivotal functors preserve traces and therefore dimensions and the spherical pairing. As elements of $\mathbb{C} \cong \mathcal{C}(I_\mathcal{C}, I_\mathcal{C}) \cong \mathcal{D}(I_\mathcal{D}, I_\mathcal{D})$, it follows that for any endomorphism $f\colon X \to X$ the following holds:

$$\operatorname{tr}(f) = \operatorname{tr}(Ff) \tag{2.1.17}$$

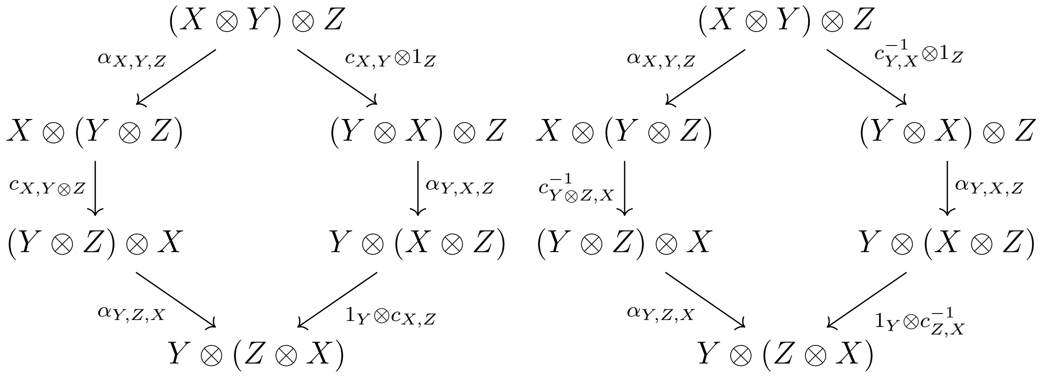
Proof. Insert the isomorphism $\mathcal{C}(I_\mathcal{C}, I_\mathcal{C}) \xrightarrow{F} \mathcal{D}(FI_\mathcal{C}, FI_\mathcal{C}) \xrightarrow{F^0\circ - \circ (F^0)^{-1}} \mathcal{D}(I_\mathcal{D}, I_\mathcal{D})$ explicitly. It is now necessary to prove $(F\operatorname{tr}(f)) \circ F^0 = F^0 \circ \operatorname{tr}(Ff)$.

$$\begin{aligned} & F\operatorname{tr}(f) \circ F^0 \\ &= F\left(\operatorname{ev}_X \circ \left(\left(f \circ i_X^{-1}\right) \otimes 1_{X^*}\right) \circ \operatorname{coev}_{X^*}\right) \circ F^0 \\ &= F\operatorname{ev}_X \circ F^2_{X,X^*} \circ \left(\left(Ff \circ Fi_X^{-1}\right) \otimes 1_{F(X^*)}\right) \circ \left(F^2_{X^{**},X^*}\right)^{-1} \circ F\operatorname{coev}_{X^*} \circ F^0 \\ &= F^0 \circ \operatorname{ev}_{FX} \circ \left(\left(Ff \circ Fi_X^{-1} \circ u_{X^*}^{-1}\right) \otimes u_X\right) \circ \left(F^2_{X^{**},X^*}\right)^{-1} \circ \operatorname{coev}_{F(X^*)} \\ &= F^0 \circ \operatorname{ev}_{FX} \circ \left(\left(Ff \circ i_{FX}^{-1} \circ (u_X^*)^{-1}\right) \otimes u_X\right) \circ \operatorname{coev}_{F(X^*)} \\ &= F^0 \circ \operatorname{ev}_{FX} \circ \left(\left(Ff \circ i_{FX}^{-1}\right) \otimes 1_{(FX)^*}\right) \circ \operatorname{coev}_{(FX)^*} \\ &= F^0 \circ \operatorname{tr}(Ff) \end{aligned}$$

□

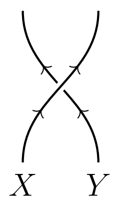
Braided, balanced and ribbon categories

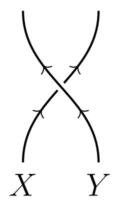
Definition 2.24. A **braided monoidal category** (or simply "braided category") is a monoidal category $\mathcal{C}$ with a dinatural isomorphism c (the "braiding") with components $c_{X,Y} \colon X \otimes Y \to Y \otimes X$ satisfying compatibility axioms with the monoidal product, called the *braid axioms*, or hexagon identities:



(2.1.18)

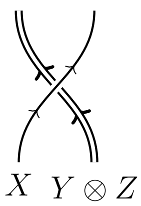
As the name suggests, the graphical calculus for braidings consists of strings which can cross each other:

$c_{X,Y} =$ 

$c^{-1}_{Y,X} =$ 

(2.1.19)

The coherence isomorphisms α are invisible in the graphical calculus. Therefore, the braid axioms become



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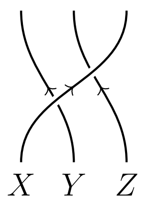

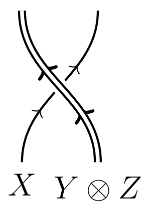

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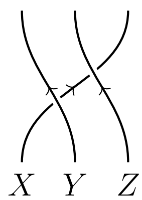


(2.1.20)

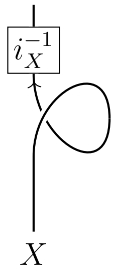
Definition 2.25. A **balanced monoidal category** is a braided category $\mathcal{C}$ with a natural isomorphism $\theta \colon 1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}}$, the **twist**, satisfying the *balance equation*:

$$\theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_Y \otimes \theta_X) \tag{2.1.21}$$

(This term should not be confused with the unrelated concept of a "balanced category", where every morphism that is mono and epi is also an isomorphism.)

Theorem 2.1. In a rigid, braided category, there exists a (noncanonical) bijection between twists satisfying the balance equation and pivotal structures. For a given pivotal structure, one possible balanced structure can be defined as:

$\theta_X :=$



(2.1.22)

For further details, consult e.g. [Sel10, Lemma 4.20], and the sources cited therein.

There are other possibilities to construct a pivotal structure from a balanced structure, but they will coincide in the case of the following definition.

Definition 2.26. A **ribbon category** is a balanced monoidal, rigid category satisfying the *ribbon equation*:

$$\theta_{X^*} = \theta_X^* \tag{2.1.23}$$

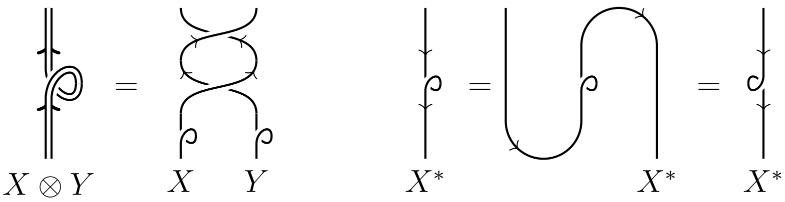
Ribbon categories are also called "tortile" categories.

The graphical representation of the twist is usually a ribbon that has been twisted by 2π. The thickening to two-dimensional ribbons is meant to express the fact that the twist cannot be undone by an ambient isotopy in three-dimensional space. In two-dimensional diagrams, ribbons can still be drawn as lines – possibly with crossings – when the blackboard framing is implicitly assumed. After recognising that the pivotal structure is a coherence and can be omitted from (2.1.22), the diagram for the twist becomes:

$\theta_X =$

(2.1.24)

The graphical representations of the balance equation and the ribbon equation are thus:



(2.1.25)

The last equality introduced the graphical representation for θ_X^*.

Definition 2.27. Ribbon fusion categories are simply ribbon categories that are also fusion categories. They are also called "premodular categories".

Remark 2.28. Ribbon categories have a canonical pivotal structure that is spherical. The spherical condition is a consequence of (2.1.23). As a partial converse, the twist of a braided spherical category is ribbon structure if it is fusion. For more details see [Dri+10, definition 2.29] and the references therein.

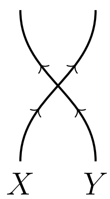
Symmetric categories

Definition 2.29. A braided category is called **symmetric** iff $c_{X,Y} = c_{Y,X}^{-1}$. A symmetric category which is also fusion is called a **symmetric fusion category**.

Remark 2.30. As a consequence of (2.1.21), a ribbon category is symmetric if the twist is trivial, although there exist symmetric ribbon categories with non-trivial twist.

If the braiding is symmetric, over- and underbraiding are set equal in the diagrammatic calculus:

$$c_{X,Y} = c_{Y,X}^{-1} = \quad (2.1.26)$$



Theorem 2.31 (After Deligne, [Del02])**.** In a symmetric fusion category, dimensions of simple objects are integers. If the twist is trivial and all dimensions are positive, then there exists a (pivotal) fibre functor to vector spaces, and the symmetric fusion category is equivalent to the representations of the finite automorphism group of the fibre functor.

Part II

Dichromatic state sum models for four-manifolds from pivotal functors

This part is based primarily on a joint article with my supervisor John Barrett [BB16].

3 Introduction

The Crane-Yetter model [CYK97] is a state sum invariant of four-dimensional manifolds that determines a topological quantum field theory (TQFT). The purpose of this part is to give a more general construction that puts the Crane-Yetter model in a wider context and allows the exploration of new models, as well as a more thorough understanding of the Crane-Yetter model itself. There is interest in four-dimensional TQFTs from solid-state physics, where they allow the study of topological insulators, for example in the framework of Walker and Wang [WW12], which is expected to be the Hamiltonian formulation of the Crane-Yetter TQFT. The Crane-Yetter model is also the starting point for constructing spin foam models of quantum gravity [BC98]. Therefore the main motivation for this paper is to provide a firmer and more unified basis for a variety of physical models.

A state sum model is a discretised path integral formulation for a lattice theory. In order to calculate the transition amplitude from one lattice state to another (possibly on a different lattice), a cobordism, or spacetime, from the initial to the final lattice is discretised using a triangulation or a cell complex. Then the amplitude is the sum of a weight function over states on the discretised cobordism. A state is typically a labelling of the elements of the discretisation with some algebraic data, for example objects and morphisms in a certain category.

In a topological state sum model, the sum over all states is independent of the particular discretisation chosen, and thus gives rise to a TQFT. The weight function corresponds to an action functional and is calculated locally, for example per simplex if the discretisation is a triangulation. This property is motivated by the physical assumption of the action being local, and is expected to have the far-reaching mathematical consequence that the resulting TQFT is 'fully extendable', which means that it is well-defined on manifolds with corners of all dimensions down to zero.

Topological state sum models are an approach for quantum gravity. The Turaev-Viro state sum is an excellent model of three-dimensional Euclidean quantum gravity ([Bar95, Section V.B] and [Bar03]). As Witten famously remarks [Wit89, Section 3], one would expect any manifestly diffeomorphism-covariant theory to give rise to a topological quantum theory. So far, no topological state sum has modelled four-dimensional quantum gravity in a satisfactory way. The most prominent topological state sum model remains the $U_q sl(2)$-Crane-Yetter state sum; however this is not considered a gravity model. It was shown to reduce to the signature [CYK97] and the Reshetikhin-Turaev theory on the boundary [BFG07]. As a consequence of this, the dimensions of the state spaces attached to the boundary manifolds are only one-dimensional, whereas in a gravity theory one would expect a large state space containing many graviton modes. The more general framework developed here suggests some different Crane-Yetter type models that may be related to approaches such as teleparallel gravity [BW12].

3.1 The Crane-Yetter invariant and its dichromatic generalisation

In three-dimensional topology, the Turaev-Viro state sum invariant distinguishes even some homotopy-equivalent three-manifolds: By [Sok97, Proposition 2], the lens spaces $L(7,1)$ and $L(7,2)$, which are homotopy equivalent, but not homeomorphic, have different values for the Turaev-Viro invariant. However the Crane-Yetter invariant of four-manifolds for *modular* categories, as it was originally defined, is just a function of the signature and the Euler characteristic of the manifold [CYK97, Proposition 6.2].

A closer look at the construction reveals a possible explanation why this is the case. By the Morse theorem, smooth manifolds admit handle decompositions. (Additionally, there is a canonical handle decomposition determined by any triangulation, by thickening the dual complex.) Different handle decompositions of the same manifold can be related by a sequence of handle slides and cancellations. Thus, one can construct a manifold invariant by assigning numbers to handle decompositions; if the numbers do not change under the handle moves, they define an invariant.

Handle decompositions can be described by Kirby diagrams. These are framed links where the components of the link represent the 1- and 2-handles. For the modular Crane-Yetter invariant, the components of the link are each labelled by the Kirby colour of the ribbon fusion category $\mathcal{C}$ that determines the invariant. By the universal property of the tangle category [Shu94], this can be interpreted as diagrammatic calculus in $\mathcal{C}$. Evaluating the diagram and multiplying by a normalisation gives the invariant.

Since the 2-handles are treated in the same way as the 1-handles, there is a redundancy in the construction of the modular Crane-Yetter invariant: it does not change if all 1-handles are replaced by 2-handles in the link diagram. But such a replacement radically changes the topology of the manifold and ensures, for example, that every manifold has the same modular Crane-Yetter invariant as a simply-connected one. Consequently, the invariant cannot even detect the first homology.

The solution is to define invariants that label the 1- and 2-handles with different objects in the category. Petit's "dichromatic invariant" [Pet08] does exactly this: in addition to the ribbon fusion category, one also chooses a full fusion subcategory and labels the 2-handles with the Kirby colour of the subcategory. Whether this change actually improves the invariant remained unstudied at the time. It will be shown in Section 8.2 that it does indeed lead to a stronger invariant that is sensitive to the fundamental group and can thus distinguish manifolds with the same signature and Euler characteristic. Now one can indeed pinpoint the improvement of the invariant as due to the differing labels on 1-handles and 2-handles. As a bonus, the general Crane-Yetter invariant is recovered as a special case of the dichromatic invariant. Previously, no description of it in terms of Kirby calculus was known for nonmodular ribbon categories.

A generalisation of the dichromatic invariant is presented here and translated into a state sum model. Instead of a ribbon fusion subcategory, the generalisation is to use a pivotal functor from a spherical fusion category to a ribbon fusion category. The 1-handles are still labelled with the Kirby colour of the target category, but the 2-handles are labelled with the Kirby colour of the source category, with the functor applied to it.

3.2 Outline

In Section 4, the graphical calculus of spherical and ribbon fusion categories is recalled. Various notational conventions are established.

In Section 5, the *sliding lemma* from spherical and ribbon fusion categories is generalised. The original lemma allows for sliding the identity morphism of any object over an encirclement by the Kirby colour of the category. The generalised lemma generalises this to an encirclement by the image of a Kirby colour under a pivotal functor. This generalisation will be a key step in the proof of invariance (Section 5.3) of the *generalised dichromatic invariant* (Definition 5.5) of smooth, oriented, closed four-manifolds. The section concludes with some general properties of the invariant and a motivating special case, Petit's dichromatic invariant (Example 5.15).

Many functors lead to the same invariant, and a general situation in which this is the case is presented in Section 6. This often leads to a simplification of the invariant, especially when the functor and both categories are unitary, or when the target category is modularisable.

If the target category of the functor is modularisable, which is often the case, the generalised invariant can also be cast in the form of a state sum. In Section 7, this state sum formula (7.2.5) is derived using the chain mail technique.

Section 8 is a non-exhaustive survey of several different examples of the generalised dichromatic invariant. The Crane-Yetter state sum is recovered as a special case, both for modular and nonmodular ribbon fusion categories. For the nonmodular Crane-Yetter invariant, a chain mail construction was not previously known. A further special case is Dijkgraaf-Witten theory without a cocycle, implying that the invariant can be sensitive to the fundamental group. The Dijkgraaf-Witten example is then generalised to group homomorphisms.

There is a discussion in Section 9 of how the present framework could connect to Walker-Wang models and state sum models used in the study of quantum gravity such as spin foam models. Relations to Cartan geometry and teleparallelism are discussed as well.

Finally, a handy overview of the different known special cases of the generalised dichromatic invariant is given as a table in Section 10, together with some comments on the results.

4 Preliminaries

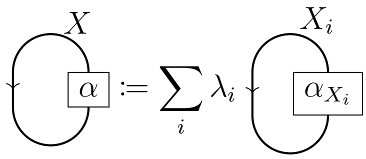
4.1 Diagrammatic calculus on spherical fusion categories

Definition 4.1. For a fusion category $\mathcal{C}$, let the **fusion algebra** $\mathbb{C}[\mathcal{C}]$ be the complex algebra generated by its objects, modulo isomorphisms and the relations $X \oplus Y = X + Y$ and $X \otimes Y = XY$.

Remark 4.2. If $\mathcal{C}$ is braided, $\mathbb{C}[\mathcal{C}]$ is commutative.

Definition 4.3. By a handy generalisation of notation, closed loops involving only (extra-)natural transformations α can also be labelled with elements of the fusion algebra, in this context called **colours**, instead of mere objects. The evaluation of a diagram with a linear combination of objects is defined as the sum of the evaluations of the diagrams with the individual objects:

$$X \coloneqq \sum_i \lambda_i X_i \tag{4.1.1}$$



(4.1.2)

Since braiding and twist are natural transformations, colours can be used in the diagrammatic calculus.

Definition 4.4. The **Kirby colour** $\Omega_\mathcal{C}$ of a spherical fusion category $\mathcal{C}$ is defined as the sum over the simple objects in $\Lambda_\mathcal{C}$ weighted by their dimensions:

$$\Omega_\mathcal{C} \coloneqq \sum_{X \in \Lambda_\mathcal{C}} d(X)\, X \tag{4.1.3}$$

Its dimension $d(\Omega_\mathcal{C}) = \sum_{X \in \Lambda_\mathcal{C}} d(X)^2$ is known as the **global dimension** of the category. It is always positive, since the field $\mathbb{C}$ has characteristic zero [ENO05].

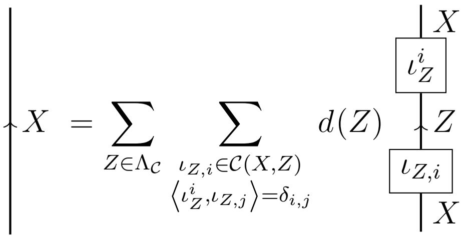
The following two lemmas are well-known, e.g., in [CYK97, Section 2].

Lemma 4.5 (Schur's lemma)**.** Any endomorphism $f\colon X \to X$ of a simple object with non-zero dimension satisfies:

$$f = 1_X \cdot \frac{\operatorname{tr}(f)}{d(X)} \tag{4.1.4}$$

Proof. Since X is simple, $\mathcal{C}(X, X) \cong \mathbb{C}$, so every endomorphism is a multiple of the identity. Taking the trace on both sides of the equation $f = \lambda 1_X$ yields the result. □

Lemma 4.6 (Insertion lemma)**.** For any object X in a spherical fusion category, its identity morphism can be decomposed into a weighted sum of identities of simple objects Z:



(4.1.5)

The $\iota_{Z,i}$ form a basis of $\mathcal{C}(X,Z)$ to which the $\iota^j_Z \in \mathcal{C}(Z,X)$ are the dual basis with respect to the spherical pairing $\langle -,- \rangle$ defined in (2.1.15).

Proof. The definition of semisimplicity 2.4 implies that

$$1_X = \sum_{Z,i} \beta^i_Z \iota_{Z,i} \tag{4.1.6}$$

for some $\beta^i_Z \in \mathcal{C}(Z,X)$. Then inserting this equality into $\left\langle \iota_{Z,j}, 1_X \iota^k_Z \right\rangle$ shows that $\beta^i_Z = d(Z)\, \iota^i_Z$. □

Remark 4.7. The insertion lemma is a generalisation of the fact from linear algebra that any vector can be decomposed uniquely into a linear combination of basis vectors.

Due to its similarity to (4.1.2), it is common to say that (the identity of) the Kirby colour $\Omega_{\mathcal{C}} = \sum_{Z\in\Lambda_{\mathcal{C}}} d(Z)\, Z$ can always be inserted in X's identity. This explains the particular name of the lemma.

Ribbon fusion categories

This subsection introduces some notation and known lemmas in ribbon fusion categories. These are also known as premodular categories.

Definition 4.8 (Graphical calculus for links)**.** Let L be an oriented framed link with a partition of its components into N sets. Choose a regular diagram of the link in the plane such that the blackboard framing from the diagram matches the original framing of the link. Given a labelling $(X_1, X_2, \ldots, X_N)$ of the sets with colours from a ribbon fusion category $\mathcal{C}$, label the link components in each set with the colour of the set and interpret the diagram as a morphism in $\mathcal{C}$, in the following way: Insert identity morphisms for vertical lines, braidings for crossings, evaluations for maxima of lines and coevaluations for minima. They are composed and tensored according to the vertical and horizontal structure of the diagram. The whole procedure is explained rigorously in [Shu94] and [Sel10].

Since a link has no open ends, the resulting morphism will be an endomorphism of I, which is essentially a complex number. This number is denoted as $\langle L(X_1, X_2, \ldots, X_N)\rangle$ and called the **evaluation** of the labelled link diagram (not to be confused with the evaluation morphisms ev_X). A labelled link diagram will sometimes be used interchangeably with its evaluation.

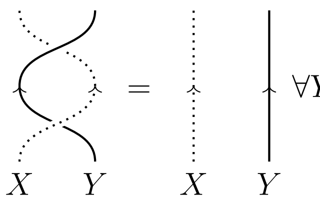
Remarks 4.9. Note that the choice of diagram for the framed link doesn't matter as two diagrams only differ by isotopies and (second and third) Reidemeister moves, which amount to identities (e.g. naturality squares or axioms like the snake identity) in the category.

It is necessary that $\mathcal{C}$ is ribbon since this ensures that the framing coefficients of the link are translated into twists of $\mathcal{C}$.

Definition 4.10. An object X is called **transparent** (or "central") in $\mathcal{C}$ if it braids trivially with any object Y in $\mathcal{C}$, that is, $c_{Y,X} \circ c_{X,Y} = 1_{X\otimes Y}$. The graphical representation of this condition is found in (4.1.7).

The full symmetric monoidal subcategory $\mathcal{C}' \subset \mathcal{C}$ with all transparent objects of $\mathcal{C}$ is called the **symmetric centre** (or "centraliser") of $\mathcal{C}$, as for example in [Mü03b] or [Dri+10]. The set of equivalence classes of simple transparent objects in $\mathcal{C}$ is then $\Lambda_{\mathcal{C}'}$. Dotted lines represent transparent objects.

$X \in \mathrm{ob}\,\mathcal{C}' \iff$



$\forall Y \in \mathrm{ob}\,\mathcal{C}$ (4.1.7)

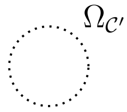
Definition 4.11. Assume X is a colour $X = \lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_N X_N$ with all X_i simple, further assume that X_1 to X_k are transparent and X_{k+1} to X_N are not. Then define the **transparent colour** $X' := \lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_k X_k + 0X_{k+1} + \cdots + 0X_N$.

Definition 4.12. The **transparent Kirby colour** is defined as follows.

$$\Omega_{\mathcal{C}'} = \Omega'_{C} = \sum_{X \in \Lambda_{\mathcal{C}'}} d(X)\, X \tag{4.1.8}$$

In the same manner, the **transparent dimension** is defined:

$$d(\Omega_{\mathcal{C}'}) =$$



$$= \sum_{X \in \Lambda_{\mathcal{C}'}} d(X)^2 \tag{4.1.9}$$

Definition 4.13. A category is called **modular** if it has $\Lambda_{\mathcal{C}'} = \{I\}$, i.e. the monoidal identity I is the only transparent object.

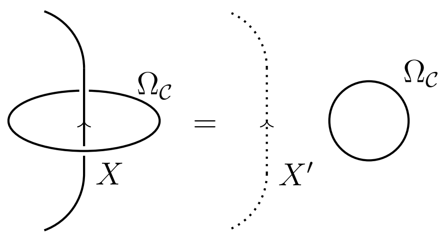
The transparent dimension of a modular category is therefore 1. Note that the multifusion case, where I is not a simple object, is excluded here.

Remark 4.14. An object that is not transparent in $\mathcal{C}$ can still be transparent in a subcategory $\mathcal{B} \subset \mathcal{C}$.

Encirclement

The technique of encirclement allows for many elegant and powerful calculations. It is indispensable when defining invariants derived from ribbon fusion categories and Kirby diagrams. Its power comes from the so-called *killing property*. This is also known as the Lickorish encircling lemma [Lic93], see also [Rob95]. It can be generalised from modular to ribbon fusion categories [Bru00, Lemma 1.4.2, in different notation].

Lemma 4.15 (Killing property)**.** In a ribbon fusion category, the following holds for any object:



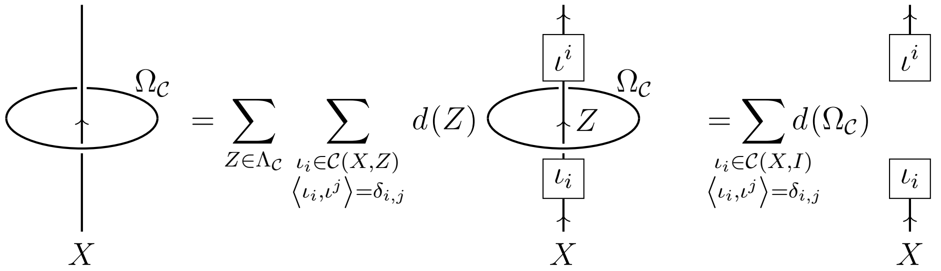
(4.1.10)

Let in particular X be simple. Then $X' = X$ if it is transparent, and 0 otherwise. In the latter case one says that X is "killed off".

Note that the orientation for the circle containing $\Omega_{\mathcal{C}}$ does not need to be specified since the colour is self-dual.

The combination of the killing property 4.15 and the insertion lemma 4.6 gives the explicit morphism of an arbitrary object encircled with the Kirby colour.

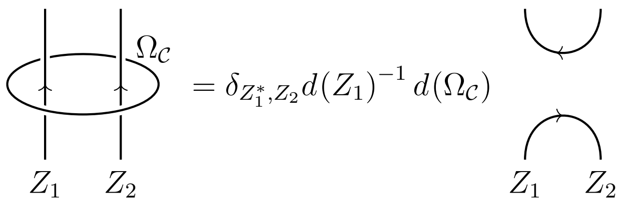
Lemma 4.16 (Cutting strands)**.** Let X be an arbitrary object of a modular category $\mathcal{C}$. Then:



(4.1.11)

The last step uses the fact that in a modular category, I is the only transparent object.

Lemma 4.17 (Cutting two strands)**.** Let Z_1, Z_2 be simple objects of a modular category $\mathcal{C}$. Then as a special case of the previous lemma:



(4.1.12)

To see the prefactors, observe that $\mathcal{C}(Z_1 \otimes Z_2, I) \cong \mathcal{C}(Z_2, Z_1^*)$. This is isomorphic to $\mathbb{C}$ if $Z_1^* \cong Z_2$, and 0 otherwise. If $Z_1^* \cong Z_2$, then $\mathcal{C}(Z_1 \otimes Z_2, I)$ is spanned by ev_{Z_1}. Since $\widetilde{\mathrm{coev}}_{Z_1} \circ \mathrm{ev}_{Z_1} = d(Z_1)$, the dual basis element must be $\widetilde{\mathrm{coev}}_{Z_1} \cdot d(Z_1)^{-1}$.

4.2 4-Manifolds and Kirby calculus

An extensive treatment of these topics is found in [GS99], [Akb16] and [Kir89]. The essential definitions and facts are highlighted here.

Handle decompositions

Let D^k denote the closed k-disk, or k-ball. The space $D^k \times D^{4-k}$, $k \in \{0, 1, 2, 3, 4\}$, is called a 4-dimensional k-handle. All handles have the same underlying topological space, but they differ in the way they are attached to each other. The boundary of a k-handle is $\partial \left(D^k \times D^{4-k}\right) = S^{k-1} \times D^{4-k} \cup D^k \times S^{3-k}$, where $S^{-1} = \emptyset$. The first component of the boundary is called the **attaching boundary** or “attaching

k	Space	Attaching boundary	Remaining boundary
0	$D^0 \times D^4$	$\emptyset$	$S^3 \cong \mathbb{R}^3 \cup \{\infty\}$
1	$D^1 \times D^3$	$S^0 \times D^3 \cong \{-1, 1\} \times D^3$	$D^1 \times S^2 \cong [-1, 1] \times S^2$
2	$D^2 \times D^2$	$S^1 \times D^2$	$D^2 \times S^1$

Table 4.1: Some relevant special cases of 4-dimensional k-handles and their boundaries.

region", the second component the **remaining boundary** or "remaining region". Some examples are shown in Table 4.1.

Smooth manifolds admit handle decompositions. A k-handle can be attached to a manifold with boundary by embedding its attaching region into the boundary of the manifold. A k-handlebody is obtained by attaching a disjoint union of k-handles to a $k-1$ handlebody, and is thus a union of 0-, 1-, ... and k-handles. Note that 0-handles have no attaching region, and a 0-handlebody is just a disjoint union of 0-handles, which are D^4s. Every n-manifold can be decomposed into handles, that is, it is diffeomorphic to an n-handlebody.

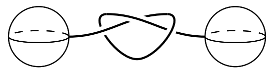
The handle decomposition is by no means unique. Two handle decompositions of diffeomorphic manifolds are always related by "handle moves", which are either cancellations of a k- and a $(k+1)$-handle, or a slide of a $(k+m)$- over a k-handle.

For a connected manifold it is always possible to arrive at a handle decomposition with exactly one 0-handle by cancelling 0-1-handle pairs. Similarly, for a closed connected n-manifold it is always possible to have exactly one n-handle by cancelling $(n-1)$-n-handle pairs.

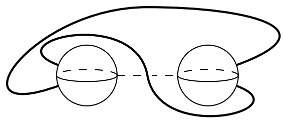
Kirby diagrams and dotted circle notation

For the 2-handlebody of a four-manifold, one can specify the handles and their attaching maps by identifying the boundary of the single 0-handle with $\mathbb{R}^3 \cup \infty$ and drawing pictures of the attaching regions of the 1- and 2-handles. This is explained in [GS99, Section 5.1]. An attachment of a 1-handle amounts to choosing two 3-balls $D^3 \times \{-1, 1\} \cong D^3 \sqcup D^3 \subset \mathbb{R}^3$, which are identified by an orientation-reversing map. A 2-handle attachment is an embedding of $D^2 \times S^1$, which is, up to isotopy, a framed embedding of S^1, i.e. a framed knot. When a part of the 2-handle is attached to a 1-handle, the S^1 of the 2-handle will enter one of the 3-balls of the 1-handle and leave the other 3-ball with which the former has been identified. The diagram of the attaching regions in $\mathbb{R}^3$ is called a **Kirby diagram**. Some examples can be found

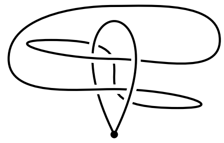
(a) A 1-handle is attached to a 0-handle by glueing the attaching boundary of the 1-handle ($\{-1, 1\} \times D^3$) to the boundary of the 0-handle ($S^3 \cong \mathbb{R}^3 \cup \{\infty\}$).



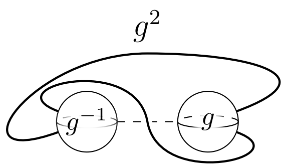
(b) A single 2-handle (possibly knotted or framed) cancels a 1-handle if it is not linked to any other handles. (The 1-handle may be linked to other handles.)



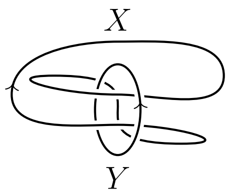
(c) A Kirby diagram of a handle decomposition of $I \times \mathbb{R}P^3$, with a single 1-handle and a single 2-handle. To convert it into an Akbulut diagram, choose a cancelling 2-handle (represented by a dashed line).



(d) In an Akbulut diagram, or special framed link, 1-handles are represented by dotted circles.



(e) A Kirby diagram gives a presentation of the fundamental group. Here, $\pi_1\left(I \times \mathbb{R}P^3\right)$ is generated by g and the relation $g^2 = 1$.



(f) An oriented Akbulut diagram can be labelled with objects from a ribbon fusion category, and subsequently interpreted in its diagrammatic calculus.

Figure 4.1: Kirby diagrams and Akbulut diagrams

in Section 8.2.

A theorem ensures that for a closed four-manifold M, specifying the 2-handlebody of a handle decomposition determines M up to diffeomorphism, i.e. any way of adding the 3- and 4-handles will yield the same manifold. Thus a closed manifold is specified uniquely (up to diffeomorphism) by its Kirby diagram.

The **dotted circle notation** for 1-handles developed by Akbulut is sometimes more convenient. Instead of adding a 1-handle, one can add a cancelling 1-2-handle pair (as shown in Figures 4.1b and 4.1c) and, after adding all further 2-handles, remove the cancelling 2-handle. In the diagram, the step of adding the cancelling pair does not require any notation because it does not change the topology. However one needs a notation to indicate how the cancelling 2-handle is removed [Kir89, Section 1.2]. Recall that a 2-handle is attached by $D^2 \times S^1 \subset D^2 \times D^2$ and so the remaining part of the boundary is $S^1 \times D^2$. This thickened (0-framed) circle is sufficient to indicate the 2-handle and is included in the diagram to represent the 1-handle it cancels. To distinguish the 1- and 2-handles, dots are drawn on those circles representing 1-handles, as in Figure 4.1d.

In Section 5.3, it will be detailed which moves one can perform on handle decompositions without changing the diffeomorphism class of the manifold. Further examples can be found in Section 8.2.

Note that the sublink consisting of only dotted circles is an unlinked union of 0-framed unknots, but 2-handle circles can be linked with each other. The 2-handle circles can also be linked with the dotted circles; this happens whenever a 2-handle runs over a 1-handle. Links of this type are called **special framed links**.

To produce an Akbulut picture from a Kirby picture [GS99, Section 5.4], take the two 3-balls of a 1-handle. The cancelling 2-handle connects them with a framed interval, or an embedding of $D^2 \times [-1, 1]$, with the ends on the 3-balls. Now instead of drawing the balls, draw the dotted circle $S^1 \times \{0\} \subset D^2 \times [-1, 1]$. A 2-handle running over this 1-handle is then drawn as a continuous line going through the dotted circle.

Definition 4.18 (Evaluation of Akbulut pictures)**.** The dotted circle notation of a handle decomposition of a closed, oriented four-manifold will be important in the definition of the invariant. The dots specify a partition of the special framed link diagram L in two sublinks, corresponding to the 1-handles and the 2-handles, respectively. After arbitrarily chosing orientations on each S^1, the two sublinks can be labelled with two colours X and Y of a ribbon fusion category. (The colours then need to be self-dual such that the chosen orientations don't matter.) Each dotted link component (1-handle) is labelled with the colour X and each of the remaining components (2-handles) is labelled with Y, and the dots can then be removed. The labelled link is denoted $L(X, Y)$. As in Definition 4.8, the evaluation is then $\langle L(X, Y) \rangle$.

Note that the relation between the two graphical notations for 1-handles mimicks the diagrammatic representation of Lemma 4.16. This will be exploited in Section 6.3, where a definition of the invariant in terms of the Kirby diagram is given.

The fundamental group

A Kirby diagram for a manifold M gives rise to a presentation of its fundamental group $\pi_1(M)$. Each 1-handle is a generator, while the 2-handles are the relations.

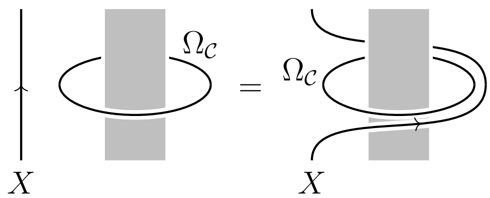
More specifically, choosing a basepoint in the 0-handle and an arbitrary direction on each 1-handle, there is a homotopy class of noncontractible curves going through a 1-handle once. A 2-handle gives a way of contracting the S^1 on its own attaching region, which is drawn in the Kirby diagram. Thus the composition of the curves going through the 1-handles along which the 2-handle is attached can be equated with the contractible curve.

This can be visualised as follows. Each 1-handle is associated to a generator. One of its corresponding 3-balls is labelled with the generator and the other with its inverse, thus fixing a direction on the 1-handle. For every circle coming from a 2-handle, choose an orientation and construct a word of generators by going once along the circle, writing down the generator (or its inverse) when entering a 1-handle through a 3-ball. (No action needs to be taken when leaving a ball.) The resulting word is then a relation in the presentation of the fundamental group. An example is given in Figure 4.1e.

5 The generalised dichromatic invariant

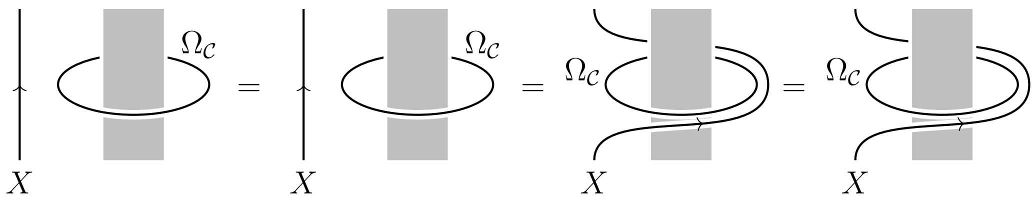
5.1 The generalised sliding property

Lemma 5.1. In a ribbon fusion category $\mathcal{C}$, the *sliding property* (in its original form due to Lickorish [Lic93]) holds:



(5.1.1)

Proof.



The killing property 4.15 has been used twice. □

As the diagrams suggest, the sliding property will later ensure that the invariant doesn't change under handle slides. To label 2-handles differently from 1-handles, it is necessary to generalise the sliding property of Lemma 5.1 to ensure invariance under the 2-2-handle slide. The idea will be to label the 2-handles with $F\Omega_\mathcal{C}$, where F is a suitable functor. Then encirclements with $F\Omega_\mathcal{C}$ must also satisfy a sliding property.

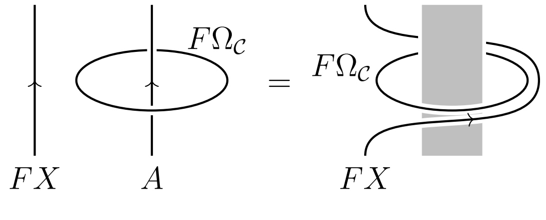
Lemma 4.6, which states that the Kirby colour can be inserted into the identity of any object, can be generalised.

Lemma 5.2 (Generalised insertion lemma)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pivotal functor, and X an object in $\mathcal{C}$. Then the identity of FX decomposes over $F\Omega_\mathcal{C} = \bigoplus_X d(X)\, FX$. In this situation, we say that $F\Omega_\mathcal{C}$ can be "inserted" into the identity of FX.

Proof. Apply F to both sides of (4.1.5) in the insertion lemma. Since pivotal functors preserve traces, they also preserve (categorical) dimensions and dual bases. □

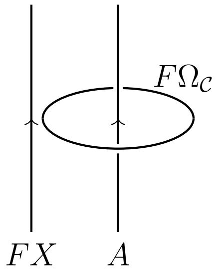
The sliding property can also be generalised in a similar way.

Lemma 5.3 (Generalised sliding property)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pivotal functor from a spherical fusion category to a ribbon fusion category. Then the following generalisation of the sliding property holds for all objects $X \in \operatorname{ob}\mathcal{C}, A \in \operatorname{ob}\mathcal{D}$:

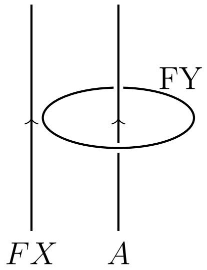


(5.1.2)

Proof. The proof proceeds diagrammatically.

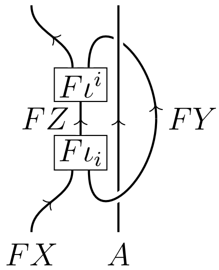


$$= \sum_{Y\in\Lambda_{\mathcal{C}}} d(Y)$$



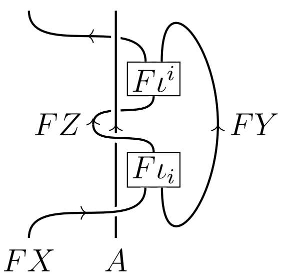
Using the definition of $\Omega_{\mathcal{C}}$. (5.1.3)

$$= \sum_{\substack{Y,Z\in\Lambda_{\mathcal{C}} \\ \iota_i\in\mathcal{C}(Z,X\otimes Y^*) \\ \langle \iota^i,\iota_j\rangle=\delta_{i,j}}} d(Y)\, d(Z)$$



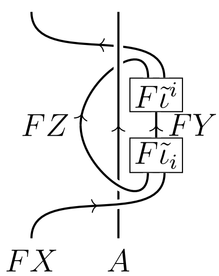
Insertion of $F\Omega_{\mathcal{C}}$, according to Lemma 5.2. (5.1.4)

$$= \sum_{\substack{Y,Z\in\Lambda_{\mathcal{C}} \\ \iota_i\in\mathcal{C}(Z,X\otimes Y^*) \\ \langle \iota^i,\iota_j\rangle=\delta_{i,j}}} d(Y)\, d(Z)$$



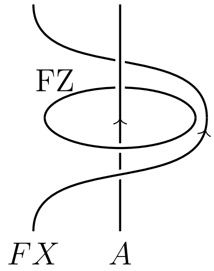
Naturality of the braiding as isotopy. (5.1.5)

$$= \sum_{\substack{Y,Z\in\Lambda_{\mathcal{C}} \\ \tilde{\iota}_i\in\mathcal{C}(Y,Z^*\otimes X) \\ \langle \tilde{\iota}^i,\tilde{\iota}_j\rangle=\delta_{i,j}}} d(Y)\, d(Z)$$



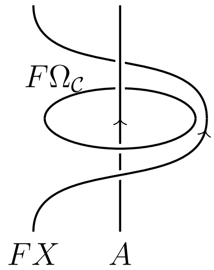
Isomorphism $\iota_i \mapsto \tilde{\iota}_i$ between $\mathcal{C}(Z, X\otimes Y^*)$ and $\mathcal{C}(Y, Z^*\otimes X)$ given by composition with evaluation and coevaluation. The $\tilde{\iota}$ form dual bases because of pivotality of F and sphericality of $\mathcal{D}$. (Explained below.) (5.1.6)

$$= \sum_{Z \in \Lambda_{\mathcal{C}}} d(Z)$$



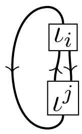
Inverse insertion of $F\Omega_{\mathcal{C}}$. Note that the two $F\Omega_{\mathcal{C}}$'s have swapped roles during the process. (5.1.7)

$$=$$



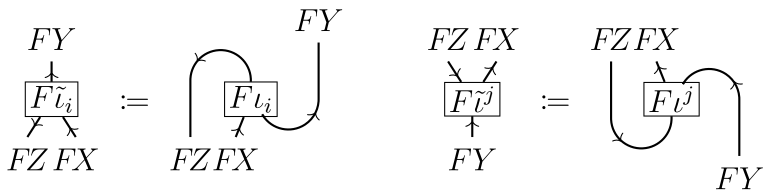
(5.1.8)

The non-obvious part of the calculation is (5.1.6). The assumption that $\{\iota_i\}$ and $\{\iota^i\}$ form dual bases with respect to the spherical pairing looks like this in the graphical calculus:



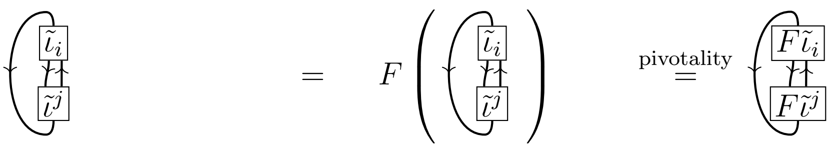
$$= \delta_{i,j} \tag{5.1.9}$$

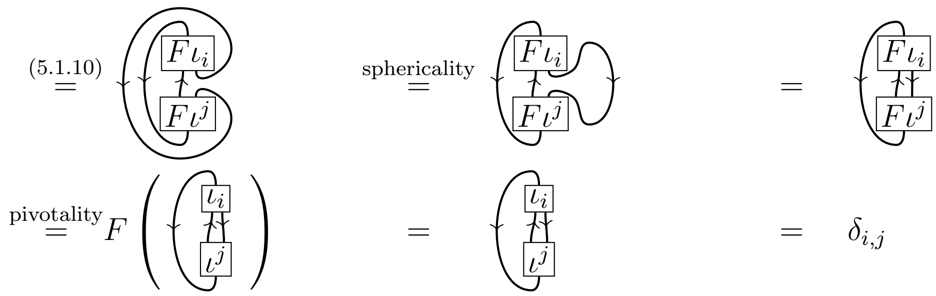
It is necessary to show that this property is also true for $\{\tilde{\iota}_i\}$ and $\{\tilde{\iota}^j\}$. After composing the $F\iota_i$ and $F\iota^j$ with evaluations and coevaluations, this again results in F applied to morphisms $\{\tilde{\iota}_i\}$ and $\{\tilde{\iota}^j\}$ since F is monoidal and therefore preserves duals (up to the natural isomorphism F^2 which is implicit here):



(5.1.10)

It is necessary to show now that $\{\tilde{\iota}_i\}$ and $\{\tilde{\iota}^j\}$ are dual bases again. But this follows from pivotality of F (preservation of traces) and sphericality of $\mathcal{D}$:

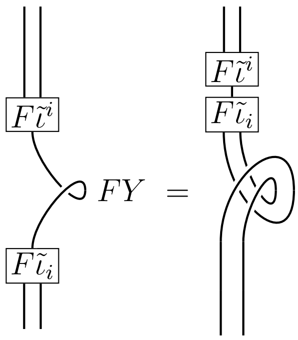




(5.1.11)

In words, pivotal functors preserve dual bases (with respect to the spherical pairing).

As is necessary, the lemma holds also if the encircling is not an unknot: Braidings and twists are natural transformations and can therefore be pushed past the $F\tilde{\iota}_i$, so they will be passed on to the new encircling morphism and the slid handle. Assume e.g. that after (5.1.6), there still is a twist on FY. Then the right hand side of the diagram is:



(5.1.12)

This argument can be easily generalised to braidings. □

Remark 5.4. It is remarkable that it is not necessary to demand $\mathcal{C}$ is ribbon, neither that F is braided or ribbon. In fact, the proof this lemma stems from a better-known sliding lemma in spherical fusion categories used for example in understanding the Hilbert spaces assigned to surfaces in the Turaev-Viro-Barrett-Westbury-TQFT [Kir11, Corollary 3.5].

For F the identity of a ribbon fusion category, the sliding lemma would have followed directly from the killing property, as demonstrated in Lemma 5.1. But if F is not the identity, it is unclear whether there is an analogue to the killing property.

5.2 The definition

The definition of the generalised dichromatic invariant can now be given.

Definition 5.5. Assume the following:

- Let $\mathcal{C}$ be a spherical fusion category.
- Let $\mathcal{D}$ be a ribbon fusion (premodular) category with trivial twist on all transparent objects.
- Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pivotal functor.
- Let L be the special framed link obtained from a handlebody decomposition of a smooth, oriented, closed four-manifold M.

Then the **generalised dichromatic invariant** of L associated with F is defined as:

$$I_F(L) := \frac{\langle L\,(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}})\rangle}{d(\Omega_{\mathcal{C}})^{h_2-h_1}\,\big(d(\Omega_{\mathcal{D}})\,d\big((F\Omega_{\mathcal{C}})'\big)\big)^{h_1}} \tag{5.2.1}$$

Here, h_i is the number of i-handles of the handle decomposition, or, the number of components in the first, respective, second set of the special framed link. $\langle L\,(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}})\rangle$ is the evaluation of the special framed link diagram as an endomorphism of $I_{\mathcal{D}}$, or equivalently a complex number, as in Definition 4.18. The 1-handles are labelled with $\Omega_{\mathcal{D}}$, the 2-handles with $F\Omega_{\mathcal{C}}$. $(F\Omega_{\mathcal{C}})'$ is the transparent part of $F\Omega_{\mathcal{C}}$, as in Definition 4.11.

Throughout, $I_F(M)$ is written instead of $I_F(L)$. This will be justified in the next subsection, where it is shown that I_F does not depend on the choice of L and is in fact an invariant of the manifold M.

Remark 5.6. It might be counter-intuitive that the unknotted, 0-framed, unlinked 1-handles are labelled by $\Omega_{\mathcal{D}}$, while the 2-handles are labelled by $F\Omega_{\mathcal{C}}$, but $\mathcal{D}$ is the ribbon category (which has algebraic counterparts of knots and framings) and $\mathcal{C}$ is only spherical. But this is indeed a valid definition, while a functor in the other direction does not lead to an invariant in an obvious way.

Note also that $F\Omega_{\mathcal{D}}$ does not depend on the monoidal coherences F^2 and F^0. Two functors with different coherences will give the same invariant. Furthermore, any two isomorphic functors will also yield the same invariant.

From now on, the conditions in the definition will be assumed, unless stated otherwise.

5.3 Proof of invariance

Lemma 5.7 (Multiplicativity under disjoint union)**.** For two links L_1 and L_2, I_F is multiplicative under disjoint union $\sqcup$:

$$I_F(L_1 \sqcup L_2) = I_F(L_1) \cdot I_F(L_2) \tag{5.3.1}$$

Proof. Evaluation of the graphical calculus is multiplicative under disjoint union: A link corresponds to an endomorphism of $\mathbb{C}$, so two links correspond to an endomorphism of $\mathbb{C} \otimes \mathbb{C}$. The evaluation is a monoidal functor with coherence isomorphism $- \cdot -: \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, so the numerator of I_F is multiplicative. Obviously, $h_i(L_1 \sqcup L_2) = h_i(L_1) + h_i(L_2)$, so the denominator is multiplicative as well. □

Given two different handle decompositions of a manifold can be transformed into each other by a series of handle slides and handle cancellations, as described for example in [GS99, Theorem 4.2.12]. The relevant moves for link diagrams of four-manifolds have been studied in [Sá79] and are explained further in [GS99, Section 5.1]. They are shown in Table 5.1.

Theorem 5.8 (Independence of handlebody decomposition)**.** The generalised dichromatic invariant is independent of the handlebody decomposition and is thus an invariant of smooth four-manifolds.

Proof. It is only necessary to check invariance of I_F under each of the handle moves in order to prove the theorem.

- Invariance under the 1-1-handle slide and the 2-1-handle slide are ensured by the sliding property 5.1. Since 1- and 2-handles are labelled with objects in $\mathcal{D}$, they can slide over a 1-handle which is labelled with $\Omega_{\mathcal{D}}$.
- Invariance under the 2-2-handle slide is ensured by the generalised sliding property 5.3. Every object in the image of F can slide over $F\Omega_{\mathcal{C}}$, so since 2-handles are labelled with $F\Omega_{\mathcal{C}}$, they can slide over each other.
- The 1-2-handle cancellation leaves I_F invariant because of its normalisation. Assume that there is a linked pair of a 1-handle and a 2-handle that is not linked to the rest of the diagram. Then it will be shown that I_F does not change if the pair is removed from the diagram. The 2-handle can be knotted, as is illustrated here with a trefoil knot. Since I_F is multiplicative under disjoint

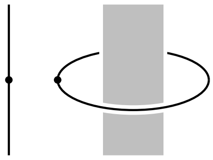
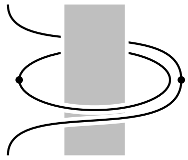
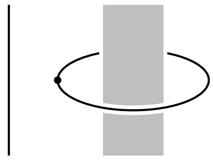
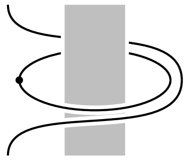
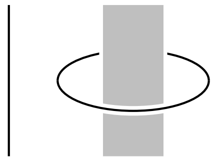
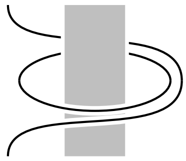
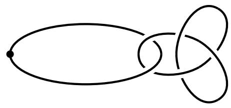
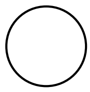
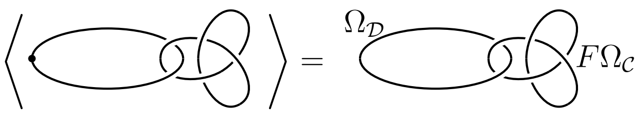
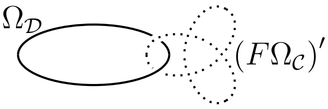
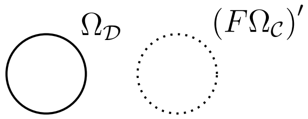
Handle move	before	after
1-1-handle slide		
2-1-handle slide		
2-2-handle slide		
1-2-handle cancellation		(empty)
2-3-handle cancellation		(empty)

Table 5.1: Handle moves and cancellations for 4-handlebodies. As usual, a dot denotes a 1-handle. The grey area stands for an arbitrary number of 1- and 2-handles passing through. Note, that for the 1-2-handle cancellation, the 2-handle may be knotted arbitrarily, but not linked to other handles. In the 2-2-handle slide, the 2-handle on the right hand side can be arbitrarily knotted, in which case the sliding handle needs to follow the blackboard framing.

union of link diagrams, it only remains to show that the invariant of the pair of handles evaluates to 1. The numerator is just the evaluation of the graphical calculus:



$=$ 

$=$ 

$$= d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big) \tag{5.3.2}$$

The number of 1-handles and 2-handles are both 1, so the denominator equals the above expression, thus the invariant is 1.

Note that it was necessary here to demand that the twist is trivial on transparent objects.

- Invariance under the 2-3-handle cancellation is even easier to show: Since there is a canonical way to attach 3- and 4-handles, they don't appear in the link picture. A 2-3-handle cancellation thus amounts to the removal of an unlinked, unknotted 2-handle. By a similar argument as before, one can evaluate the invariant on the link diagram of such a 2-handle and find that it is 1 as well.

□

Remark 5.9. Pivotality of F is essential for the invariance of I_F. As an easy counterexample, take the category of super vector spaces, which is defined as follows. As monoidal category, choose the category of finite dimensional representations of $\mathbb{Z}_2$. Choose the pivotal structure such that the sign representation σ has dimension -1, and the trivial twist. The braiding is then required to be $c_{\sigma,\sigma}(v \otimes w) = -w \otimes v$, and all other braidings trivial.

There is an obvious forgetful strong monoidal functor U to vector spaces sending both simple objects to $\mathbb{C}$. This functor is *not* pivotal since the dimension of $\mathbb{C}$ is $+1$. One finds that the evaluation of the (undotted) unknot is

$$\begin{aligned} d\big(U\Omega_{\mathrm{Rep}(\mathbb{Z}_2)}\big) &= \sum_{X \in \Lambda_{\mathrm{Rep}(\mathbb{Z}_2)}} d(X)\, d(UX) \\ &= 1 \cdot 1 + (-1) \cdot 1 = 0 \end{aligned}$$

However, the corresponding manifold is S^4 and the empty diagram (which would result from cancelling the single 2-handle with a 3-handle) evaluates to 1. It is apparent now that a non-pivotal functor can break invariance.

5.4 Simply-connected manifolds and multiplicativity under connected sum

As was shown in Lemma 5.7, the generalised dichromatic invariant is multiplicative under disjoint union of link diagrams. This operation, in turn, corresponds to connected sum of manifolds. As a consequence, for two manifolds M^1 and M^2, the invariant satisfies $I_F(M^1 \# M^2) = I_F(M^1) \cdot I_F(M^2)$, where $\#$ denotes connected sum. This has far reaching consequences, as is shown in the following known lemma.

Lemma 5.10. Assume I is any invariant of oriented, closed four-manifolds that is multiplicative under connected sum on simply-connected manifolds. Furthermore, assume that $I\left(\mathbb{CP}^2\right)$ and $I\left(\overline{\mathbb{CP}}^2\right)$ are invertible. Then I is given on a *simply-connected* four-manifold M by

$$I(M) = \left(I\left(\mathbb{CP}^2\right) I\left(\overline{\mathbb{CP}}^2\right)\right)^{-1+\frac{\chi(M)}{2}} \left(\frac{I\left(\mathbb{CP}^2\right)}{I\left(\overline{\mathbb{CP}}^2\right)}\right)^{\frac{\sigma(M)}{2}} \tag{5.4.1}$$

χ and σ are Euler characteristic and signature, respectively.

Proof. The first, and by Poincaré duality third, homologies of M are trivial, so the Euler characteristic $\chi(M)$ is equal to $2 + b_2(M)$, where $b_2(M) = b_2^+(M) + b_2^-(M)$ is the rank of the second homology and $b_2^\pm(M)$ are the dimensions of the subspaces on which the intersection form is positive or negative. Since the signature is $\sigma(M) = b_2^+(M) - b_2^-(M)$, then it follows that $b_2^\pm(M) = (\chi(M) \pm \sigma(M))/2 - 1$.

But it is well-known [GS99, Corollary 9.1.14] that simply-connected manifolds stably decompose into $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ under connected sum, i.e. there exist natural numbers m, n^+, n^- such that:

$$M \#^m \mathbb{CP}^2 \#^m \overline{\mathbb{CP}}^2 \cong \#^{n^+} \mathbb{CP}^2 \#^{n^-} \overline{\mathbb{CP}}^2 \tag{5.4.2}$$

($M \#^n N$ denotes the connected sum of M and n copies of N.) By comparing the intersection forms on both sides, one sees that that the numbers of positive and negative eigenvalues are $b_2^\pm(M) = n^\pm - m$. Therefore by multiplicativity under connected sum:

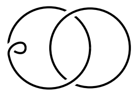
$$\begin{aligned} I(M) I\left(\mathbb{CP}^2\right)^m I\left(\overline{\mathbb{CP}}^2\right)^m &= I\left(\mathbb{CP}^2\right)^{n^+} I\left(\overline{\mathbb{CP}}^2\right)^{n^-} \\ \implies I(M) &= I\left(\mathbb{CP}^2\right)^{b_2^+(M)} I\left(\overline{\mathbb{CP}}^2\right)^{b_2^-(M)} \end{aligned}$$

Now (5.4.1) follows easily. □

Remark 5.11. Such invariants cannot distinguish the homotopy-inequivalent manifolds $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{CP}}^2$. In particular, these manifolds have different intersection forms, but the same signature. Effectively, invariants with the above properties are insensitive to this homotopical information.

Lemma 5.12. The generalised dichromatic invariant is invertible on $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$.

Proof. This is best seen by directly calculating the invariants on these manifolds. It is known that $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \cong S^2 \tilde{\times} S^2$, where the latter denotes the total space of a twisted S^2-bundle over S^2, which has the following Kirby diagram [GS99, Figure 4.34]:

$$S^2 \tilde{\times} S^2 =$$ 

To the show the invertibility of both $I\left(\mathbb{CP}^2\right)$ and $I\left(\overline{\mathbb{CP}}^2\right)$, calculate the following:

$$\begin{aligned} I\left(\mathbb{CP}^2\right) \cdot I\left(\overline{\mathbb{CP}}^2\right) &= I\left(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2\right) \\ &= \frac{\langle L_{S^2 \tilde{\times} S^2}\left(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}}\right)\rangle}{d(\Omega_{\mathcal{C}})^{h_2 - h_1}\left(d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big)\right)^{h_1}} \end{aligned}$$

The killing property 4.15 and the handle numbers $h_1 = 0$, $h_2 = 2$ give:

$$= \frac{d(F\Omega_{\mathcal{C}}) \sum_{X \in \Lambda_{\mathcal{C}}} \operatorname{tr}\left(\theta_{(FX)'}\right)}{d(\Omega_{\mathcal{C}})^2}$$

Recall that the twist is required to be trivial on transparent objects in $\mathcal{D}$. Furthermore, F is pivotal and preserves quantum dimensions.

$$= \frac{d((F\Omega_{\mathcal{C}})')}{d(\Omega_{\mathcal{C}})} \tag{5.4.3}$$

Since $F\Omega_{\mathcal{C}}$ contains at least the monoidal unit, the result cannot be 0. □

Corollary 5.13. Lemma 5.10 applies to the generalised dichromatic invariant.

Proof. $I\left(\mathbb{CP}^2\right) \cdot I\left(\overline{\mathbb{CP}}^2\right)$ is invertible due to the previous lemma. Multiplicativity under connected sum has already been shown in Lemma 5.7. □

It remains to calculate the invariants of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ in order to be able to give concrete values for simply-connected manifolds. $\mathbb{CP}^2$ can be composed of a 0-handle, a 2-handle and a 4-handle. A link diagram for it is given by an unknotted circle with framing +1, denoted by L_{+1}. The value of the invariant is therefore:

$$I\left(\mathbb{CP}^2\right) = \frac{\langle L_{+1}\left(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}}\right)\rangle}{d(\Omega_{\mathcal{C}})^{h_2-h_1}\left(d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big)\right)^{h_1}} = \frac{\sum_{X\in\Lambda_{\mathcal{C}}} \operatorname{tr}\left(\theta_{FX}\right)}{d(\Omega_{\mathcal{C}})} \tag{5.4.4}$$

Analogously:

$$I\left(\overline{\mathbb{CP}}^2\right) = \frac{\sum_{X\in\Lambda_{\mathcal{C}}} \operatorname{tr}\left(\theta_{FX}^{-1}\right)}{d(\Omega_{\mathcal{C}})} \tag{5.4.5}$$

For many cases of F, more concrete values can be calculated. This is done in Section 8.1.

5.5 Petit's dichromatic invariant and Broda's invariants

Broda defined two invariants of four-manifolds using the category of tilting modules for $U_q sl(2)$ at a root of unity [Bro93; Rob95]. The original invariant, called here the **Broda invariant**, labelled both 1- and 2-handles with simple objects in this category (the "spins"), whereas the **refined Broda invariant** labelled 2-handles with just the integer spins. The Broda invariants were investigated by Roberts [Rob95; Rob97], who showed that the Broda invariant depends on the signature of the four-manifold whereas the refined Broda invariant detects also the first Betti number with $\mathbb{Z}_2$ coefficients, and is sensitive to the second Stiefel-Whitney class (which decides whether the manifold admits a spin structure).

Generalising Broda's constructions to other ribbon fusion categories leads to the following two classes of examples.

Example 5.14. As noted by Petit [Pet08, Remark 4.4], for the identity functor $\mathbb{1}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ one recovers, up to a factor depending on the Euler characteristic, a **generalised Broda invariant** for a ribbon fusion category $\mathcal{D}$ satisfying the conditions of Definition 5.5. Petit shows that this invariant depends only on the signature (and Euler characteristic) of the four-manifold.

Example 5.15. The refined Broda invariant, which will be discussed again in Section 8.2, can be generalised to arbitrary ribbon fusion subcategories.

Let $\mathcal{C}$ and $\mathcal{D}$ be ribbon fusion categories. ($\mathcal{D}$ is not required to be modular.) For a full ribbon inclusion functor $F\colon \mathcal{C} \hookrightarrow \mathcal{D}$, **Petit's dichromatic invariant** I_0 [Pet08, (4.4)] is recovered, again up to a factor depending on the Euler characteristic $\chi(M)$, which will be calculated in the following.

The notation in [Pet08] is subtly different: $\mathcal{C}'$ denotes an arbitrary subcategory there, not necessarily the symmetric centre. Also, the notation for categorical dimensions is different from this presentation. Redefining the symbols from [Pet08] in the notation established here gives $\Delta_{\mathcal{C}} := d(\Omega_{\mathcal{C}})$ and $\Delta''_{\mathcal{D},\mathcal{C}} := d\big((F\Omega_{\mathcal{C}})'\big)$.

The nullity of the linking matrix of the link diagram has to be introduced, but since M is closed, it equals h_3, the number of 3-handles. Petit's invariant is then in the present notation:

$$I_0(L) := \frac{\langle L\left(\Omega_{\mathcal{D}}, \Omega_{\mathcal{C}}\right)\rangle}{d(\Omega_{\mathcal{C}})^{h_3} \left(d(\Omega_{\mathcal{D}})\, d((F\Omega_{\mathcal{C}})')\right)^{\frac{h_1+h_2-h_3}{2}}} \tag{5.5.1}$$

Note that the numerators of I_F and I_0 do not differ, but the normalisations do. To compare the normalisation of invariants, their ratio is calculated using a handle decomposition with exactly one 0-handle and 4-handle.

The ratio of invariants is then

$$\begin{aligned}\frac{I_F(M)}{I_0(M)} &= \frac{d(\Omega_{\mathcal{C}})^{h_3} \cdot \big(d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big)\big)^{\frac{h_1+h_2-h_3}{2}}}{d(\Omega_{\mathcal{C}})^{h_2-h_1} \cdot \big(d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big)\big)^{h_1}} \\ &= \left(\frac{\sqrt{d(\Omega_{\mathcal{D}})\, d\big((F\Omega_{\mathcal{C}})'\big)}}{d(\Omega_{\mathcal{C}})}\right)^{\chi(M)-2}\end{aligned} \tag{5.5.2}$$

The same calculation can be used to show that the refined Broda invariant from [Bro93] is Petit's invariant I_0 for the subcategory of integer spins in the category of tilting modules of $U_q sl(2)$.

Remark 5.16. Whenever a full inclusion into a ribbon category is encountered, it will be assumed that the subcategory inherits braiding and ribbon structures from the bigger category. Also, it will be assumed that the canonical pivotal structure is chosen on both sides, which is then automatically preserved.

Remark 5.17. Petit called his invariant "dichromatic" since the special framed link arising from the handle decomposition is labelled with two different Kirby colours. The invariant presented here uses two different colours as well, so it seems appropriate to keep the name "dichromatic", but to point out that it is somewhat more general.

6 Simplification of the invariant

Here it is shown that a general argument allows the generalised dichromatic invariant to be simplified in many cases.

Proposition 6.1. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \xrightarrow{H} \mathcal{D}$ be a chain of pivotal functors on spherical fusion categories. Let furthermore $\mathcal{C} \xrightarrow{H} \mathcal{D}$ be ribbon, and let the symmetric centres (Definition 4.10) $\mathcal{C}'$ and $\mathcal{D}'$ have trivial twist. Assume these three conditions on F and H, for some $m, n \in \mathbb{C}$:

$$F\Omega_{\mathcal{A}} = n\Omega_{\mathcal{B}}$$
$$H\Omega_{\mathcal{C}} = m\Omega_{\mathcal{D}}$$
$$H\left((G\Omega_{\mathcal{B}})'\right) = (HG\Omega_{\mathcal{B}})'$$

Then $I_{HGF} = I_G$.

Proof. Note that since F and H are pivotal, the values of m and n can be inferred by taking the dimensions on each side of the first two conditions:

$$d(\Omega_{\mathcal{A}}) = n \cdot d(\Omega_{\mathcal{B}})$$
$$d(\Omega_{\mathcal{D}}) = m^{-1} \cdot d(\Omega_{\mathcal{C}})$$

Let now L be a special framed link for the four-manifold M.

$$\begin{aligned}\langle L\left(\Omega_{\mathcal{D}}, HGF\Omega_{\mathcal{A}}\right)\rangle &= \langle L\left(H\Omega_{\mathcal{C}}, HG\Omega_{\mathcal{B}}\right)\rangle \cdot m^{-h_1} n^{h_2} \\ &= \langle L\left(\Omega_{\mathcal{C}}, G\Omega_{\mathcal{B}}\right)\rangle \cdot m^{-h_1} n^{h_2}\end{aligned}$$

The first two assumptions were inserted, then it was used that H is ribbon, to arrive at the enumerator of $I_F(M)$ up to the factors of m and n. Using the first and the third assumption, the missing part in the denominator of $I_F(M)$ can be calculated:

$$d\big((HGF\Omega_{\mathcal{A}})'\big) = n \cdot d\big((HG\Omega_{\mathcal{B}})'\big) = n \cdot d\big(H\,(G\Omega_{\mathcal{B}})'\big) = n \cdot d\big((G\Omega_{\mathcal{B}})'\big)$$

It is easy to see now that all factors of n and m cancel. □

In the following, it is shown that there is an abundance of functors satisfying these conditions, allowing a simplification of the generalised dichromatic invariant in many cases. Examples include cases where either H or F is the identity functor.

6.1 Simplification for unitary fusion categories

One case, in which the generalised dichromatic invariant simplifies to Petit's dichromatic invariant is the case of unitary fusion categories, which are certain non-degenerate $\mathbb{C}$-linear †-categories. The unitarity condition is important in mathematical physics, and many examples are known. The theory of unitary fusion categories is well developed, and many important properties are found in the literature, e.g. [Dri+10]. Instead of giving a self-contained introduction, the relevant known facts are listed.

- A fusion †-category with a rigid structure has a canonical spherical structure (see [Sel10, Lemma 7.5]) defined by the †-structure and the chosen duals.
- A unitary functor, or †-functor, is a functor that preserves the †-structure. A strong monoidal unitary functor is pivotal, so it preserves the canonical spherical structure.

Definition 6.2. A strong monoidal functor of fusion categories $F\colon \mathcal{C} \to \mathcal{D}$ is called **dominant** if for any object $Y \in \operatorname{ob}\mathcal{D}$ there exists an object $X \in \operatorname{ob}\mathcal{C}$ such that Y is a subobject of FX. In [ENO05] these are also known as "surjective functors".

Lemma 6.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a dominant unitary functor of unitary fusion categories. Let furthermore both categories have the canonical spherical structure coming from the unitary structure. Then the following holds:

$$F\Omega_{\mathcal{C}} = \frac{d(\Omega_{\mathcal{C}})}{d(\Omega_{\mathcal{D}})}\Omega_{\mathcal{D}} \tag{6.1.1}$$

Proof. An analogous equation holds for the Frobenius-Perron dimensions [ENO05, Proposition 8.8]. In unitary fusion categories with the canonical spherical structure Frobenius-Perron dimensions and categorical dimensions coincide. □

Definition 6.4. For a strong monoidal functor F of fusion categories, define the image category $\operatorname{Im} F$ [Dri+10, Definition 2.1]. Its objects are all objects of $\mathcal{D}$ that are isomorphic to a subobject of FX, where X is any object in $\mathcal{C}$. The morphisms of $\operatorname{Im} F$ are chosen such that it is a full fusion subcategory of $\mathcal{D}$.

Lemma 6.5. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor of fusion categories. Then $F = F_2 \circ F_1$, where $F_1\colon \mathcal{C} \to \operatorname{Im} F$ is a dominant functor, and $F_2\colon \operatorname{Im} F \to \mathcal{D}$ the full inclusion from the previous definition.

Proof. By construction of the image category, F factors through it, and F restricted to $\operatorname{Im} F$ is dominant. □

Corollary 6.6. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a strong monoidal unitary functor of unitary fusion categories, and again $\mathcal{D}$ ribbon such that its symmetric centre $\mathcal{D}'$ has trivial twist. Then $I_F = I_{F_2}$, and so is equal to Petit's dichromatic invariant I_0 for the inclusion $F_2\colon \operatorname{Im} F \hookrightarrow \mathcal{D}$, multiplied by the Euler characteristic factor from (5.5.2).

Proof. Use the previous lemma to decompose F into a dominant functor and a full inclusion. By the lemma before, the dominant part satisfies the conditions of Proposition 6.1, so I_F is reduced to the invariant for the full inclusion. The fusion subcategory inherits the pivotal structure from $\mathcal{D}$. An invariant from a full inclusion is a case of Petit's dichromatic invariant, as explained in Example 5.15. □

6.2 Modularisation

This subsection considers examples that will be compared to the Crane-Yetter invariant in Section 8.

Definition 6.7. A ribbon fusion category $\mathcal{D}$ is called **modularisable** if its symmetric centre $\mathcal{D}'$ has trivial twist and dimensions in $\mathbb{N}$. For modularisable categories, there exists a faithful functor $H\colon \mathcal{D} \to \widetilde{\mathcal{D}}$ with $\widetilde{\mathcal{D}}$ modular, called the **modularisation** (also "deequivariantisation") of $\mathcal{D}$. Some standard references are [Bru00] or [Mü00].

Remarks 6.8.

- H is usually not full.
- The name "deequivariantisation" comes from thinking of $\mathcal{D}'$ as the representations of some finite group. H restricted to $\mathcal{D}'$ then plays the role of a fibre functor, while not disturbing the nontransparent objects. $\widetilde{\mathcal{D}}$ has the same objects as $\mathcal{D}$, but additional isomorphisms from any transparent object to a direct sum of Is.
- For any symmetric fusion category without twist or pivotal structure, one can choose the trivial twist $\theta_X = 1_X$. With the corresponding pivotal structure, the categorical dimensions of objects are then in $\mathbb{Z}$. Alternatively, one can choose a pivotal structure with categorical dimensions in $\mathbb{N}$, but then the twist will usually not be trivial. To adhere to the conditions in Definition 5.5, the trivial twist will always be chosen for symmetric fusion categories.

Proposition 6.9. Let $F\colon \mathcal{C} \to \mathcal{D}$ be pivotal with $\mathcal{C}$ spherical fusion and $\mathcal{D}$ modularisable. Such a functor satisfies the conditions of our invariant in Definition 5.5. Let $H\colon \mathcal{D} \to \widetilde{\mathcal{D}}$ be the modularisation functor. Then $I_F = I_{H\circ F}$.

Proof. In [Bru00, Proposition 3.7] it is stated that $H\Omega_{\mathcal{D}} \cong d(\Omega_{\mathcal{D}'})\,\Omega_{\widetilde{\mathcal{D}}}$. It is easy to check that $\widetilde{\mathcal{D}}\,(I, H\,(X')) = \widetilde{\mathcal{D}}\,(I, (HX)')$ follows from the original definition, and therefore $H\left((F\Omega_{\mathcal{C}})'\right) = (HF\Omega_{\mathcal{C}})'$ since both sides are multiples of I. Thus, Proposition 6.1 can be applied. □

Intuitively, the transparent objects on the 1-handles can be removed and don't contribute to the invariant. The modularisation H makes this explicit by sending all objects in $\mathcal{D}'$ to multiples of I.

One can make use of this fact by noting that many generalised dichromatic invariants are equal to an invariant arising from a functor into a modular category. It is necessary to demand all dimensions of simple objects in $\mathcal{D}'$ to be positive, but this is the sole restriction. In Section 7, it will be shown that invariants with a modular target category can be expressed in terms of a state sum and therefore extend to topological quantum field theories.

Remark 6.10. The modularisation H is not a full inclusion if the source $\mathcal{D}$ is not modular (and the identity otherwise). Therefore, the composition $H \circ F$ will usually not be full either, even if F is. However, in the unitary case, the following corollary is helpful.

Corollary 6.11. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a strong monoidal unitary functor of unitary fusion categories. Let also $\mathcal{D}$ be modularisable, and H the modularisation functor. Then there is a full inclusion into a modular category $G\colon\ \mathrm{Im}(H \circ F)\widetilde{D}$, and $I_F = I_G$.

Proof. From Proposition 6.9, $I_{H\circ F} = I_F$, where H is the modularisation. Therefore Corollary 6.6 can be applied to $H \circ F$. □

6.3 Cutting strands

If the target category $\mathcal{D}$ of the pivotal functor is modular, each 1-handle is labelled by $\Omega_{\mathcal{D}}$. The strands of the 2-handles going through it can be cut, using Lemma 4.16. This is the algebraic analogue of reverting from Akbulut's dotted handle notation to Kirby's original notation for handle decompositions where each 1-handle is represented by a pair of D^3s. There is now a simpler definition of the generalised dichromatic invariant, which is obtained by cutting the strands through the 1-handles.

Definition 6.12. Let K be a Kirby diagram for a handle decomposition of a smooth, closed four-manifold M. Choose orientations on the S^1 of the attaching boundary of each 2-handle, and a choice of $+$ and $-$ signs on the respective 3-balls for each 1-handle.

1. An **object labelling** is a map X from the set of 2-handles to the set of simple objects in $\mathcal{C}$. The object assigned to the i-th 2-handle is written X_i.
2. Now, for every 1-handle with 2-handles $i \in \{1, 2, \ldots N\}$ entering or leaving the ball labelled with $+$, dual bases for the morphism spaces $\mathcal{D}(FX_1 \otimes FX_2 \otimes \cdots \otimes FX_N, I)$ and $\mathcal{D}(I, FX_1 \otimes FX_2 \otimes \cdots \otimes FX_N)$ are chosen. (The objects on leaving 2-handles are dualised.)

   A **morphism labelling** for a given object labelling is a choice of basis morphism for the $+$-ball of every 1-handle, and the corresponding dual morphism on the ball labelled with $-$.
3. For a given object and morphism labelling, the evaluation of the labelling is the evaluation of the labelled diagram as a morphism in $\mathcal{D}(I, I) \cong \mathbb{C}$, multiplied with the factor $\prod_i d(X_i)$, where i ranges over all 2-handles.
4. The evaluation $\langle K(F) \rangle$ of the Kirby diagram K is the sum of evaluations over all labellings.

Proposition 6.13. Let K be a Kirby diagram for a handle decomposition of a smooth, closed four-manifold M. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a pivotal functor from a spherical fusion category to a modular category, and let n be the multiplicity of I in $F\Omega_{\mathcal{C}}$. Then the generalised dichromatic invariant is:

$$I_F(M) = \frac{\langle K(F) \rangle}{d(\Omega_{\mathcal{C}})^{h_2 - h_1} \, n^{h_1}} \tag{6.3.1}$$

Proof. Application of Lemma 4.16 to the labelled special framed link L shows:

$$\langle L(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}}) \rangle = d(\Omega_{\mathcal{D}})^{h_1} \langle K(F) \rangle$$

Since $\mathcal{D}$ is modular, $d\big((F\Omega_{\mathcal{C}})'\big) = d(nI) = n$ and the result follows. □

This proposition can be used as an alternative definition of the invariant in most cases. However to prove invariance under all handle slides, it is more convenient to refer to the original Definition 5.5.

7 The state sum model

The Crane-Yetter invariant is originally defined using a state sum model on a triangulation of a four-manifold [CYK97]. However, it was not presented as a state sum model in Section 3.1. This is possible using a reformulation of the original definition due to Roberts, as presented in [Rob95, Section 4.3]. He showed that for modular categories, the Crane-Yetter state sum CY is equal to the Broda invariant B up to a normalisation involving the Euler characteristic, through a process called "chain mail", which will be described in the following.

This is not true for nonmodular $\mathcal{C}$: As will be shown in the next section, CY and B indeed differ in this case. The nonmodular Crane-Yetter invariant arises from Petit's dichromatic invariant and does not depend only on the signature and Euler characteristic, but also at least on the fundamental group.

Previously, it wasn't known how to derive the nonmodular Crane-Yetter invariant from a handle picture. With the generalised dichromatic invariant, it is possible to do so. Through chain mail one can recover a state sum description of the generalised dichromatic invariant I_F, whenever $F\colon \mathcal{C} \to \mathcal{D}$ such that $\mathcal{D}$ is modular. So the generalised dichromatic invariant has a purely combinatorial description in terms of triangulations in that case. The nonmodular Crane-Yetter invariant will turn out to be a special case.

In general, the state sum model will be useful to understand the physical interpretation of a particular model, while the handle picture is very convenient for calculations.

7.1 The chain mail process and the generalised 15-j symbol

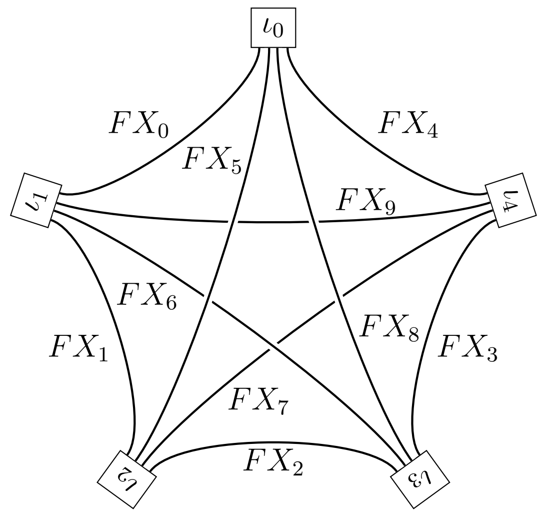
Given a four-dimensional manifold M with triangulation Δ, there is always a 0-framed, unknotted handle decomposition via the following process: Replace the triangulation by its dual complex, i.e. 4-simplices $s \in \Delta_4$ by vertices, tetrahedra $t \in \Delta_3$ by edges, triangles $\tau \in \Delta_2$ by polygons and in general $(4-k)$-simplices by k-cells. A k-cell, $k \leq 3$, will then have a valency (the number of adjacent $(k+1)$-cells) of $5-k$, coming from the number of faces of the original simplex.

Then consider the handle decomposition arising from a thickening of this dual complex. This handle decomposition has h_0 0-handles, where h_0 is then the number of 4-simplices in the triangulation, Δ_4. To work with Kirby diagrams, a decomposition with only one 0-handle is needed. In [Rob95, Section 4.3], Roberts shows that cancelling all but one 0-handle with 1-handles amounts to multiplying the invariant by $I_F(S^1 \times S^3)^{1-h_0} = d(\Omega_{\mathcal{C}})^{1-h_0}$.

Inserting morphism boxes for the two D^3s at every 1-handle arising from a tetrahedron disconnects the whole link diagram into pentagram-shaped subdiagrams for every 4-simplex.

To arrive at the pentagram shape, first realise that the boundary of a 4-simplex is $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$. This is the boundary of the 0-handle to which 1-handles and 2-handles are attached. Visualise the triangulation of the boundary by arranging four vertices of the 4-simplex as a tetrahedron around the origin and putting the remaining vertex at infinity. Connecting the first four vertices to the vertex at infinity gives the remaining four tetrahedra. Now draw one copy of D^3 for each tetrahedron (the respective copy belonging to a neighbouring 4-simplex) and connect each pair of D^3s with lines from the triangles as 2-handles. The resulting subdiagram is now a big tetrahedron of D^3s with a further D^3 in the centre of the tetrahedron. Project this subdiagram onto the plane, for every 4-simplex, and apply Definition 6.13. After applying an arbitrary isotopy in the plane, the evaluation of such a subdiagram labelled with objects $X_i \in \operatorname{ob}\mathcal{C}$ and morphisms ι_i, $i \in \{0, \ldots, 4\}$, $j \in \{0, \ldots, 9\}$ in $\mathcal{D}$ is:

$$⛤ (FX_i, \iota_i) := ⛤ (FX_0, \ldots, FX_9, \iota_0, \ldots, \iota_4)$$

:= 

(7.1.1)

The over- and under-braidings follow the convention of Roberts. It involves a "splitting convention" to arrive at a correct blackboard framing, see [Rob95, Figure 17].

The diagram does not yet correspond to a morphism. To evaluate it in terms of diagrammatic calculus of the ribbon category $\mathcal{D}$, one has to orient the lines upwards or downwards and insert evaluations and coevaluations as needed, in order to specify where an object or its dual is the source or the target of a morphism. To arrive at such a choice, fix a total ordering of the vertices. This ordering induces an orientation on the tetrahedra. Each tetrahedron occurs as the face of two 4-simplices, which are oriented as submanifolds of M, and the tetrahedron inherits two opposite orientations from each of them. Since a tetrahedron corresponds to a 1-handle, the $+$ and $-$ signs need to be distributed onto the attaching D^3s. Put the $+$ sign on the D^3 attaching to the 4-simplex from which the tetrahedron inherits the orientation agreeing with the ordering of the vertices. Consequently, its morphism is $\iota\colon X_{i_1} \otimes X_{i_2} \otimes X_{i_3} \otimes X_{i_4} \to I$, while the morphism of the other D^3 goes in the other direction.

7.2 The state sum

Since the whole diagram is a disconnected sum of diagrams of the above shape, its evaluation will be a product of ⛤-quantities. Recall Definition 4.3, where colours, such as the Kirby colour $\Omega_{\mathcal{C}}$ are understood in terms of evaluating the diagram as a sum over simple objects. This sum leads to a state sum formula for I_F. The X_i in the definition of ⛤ are then summands of $F\Omega_{\mathcal{C}}$, which was labelling the 2-handles. The ι_i label the D^3s of a 1-handle. The invariant I_F will then be a big sum over the summands of all these copies of $F\Omega_{\mathcal{C}}$ and the dual morphism bases.

Definition 7.1. An F-object labelling of the triangulation Δ is a function

$$X\colon \Delta_2 \to \Lambda_{\mathcal{C}} \tag{7.2.1}$$

For a given F-object labelling and a total ordering of the vertices Δ_0, fix bases of the morphism spaces in the following way: For every tetrahedron $t \in \Delta_3$ with vertices $v_0 < v_1 < v_2 < v_3$, denote by τ_i the face triangle of t where the vertex v_i is left out. Now choose dual bases for the space $\mathcal{D}\left(FY(\tau_0) \otimes FY(\tau_2) \otimes FY(\tau_1) \otimes FY(\tau_3), I\right)$ and its dual.

Then, using the same convention, an F-morphism labelling is a function

$$\iota\colon \Delta_3 \to \operatorname{mor} \mathcal{D} \tag{7.2.2}$$

where $\iota(t)$ is a basis vector of the space $\mathcal{D}\left(FY(\tau_0) \otimes FY(\tau_2) \otimes FY(\tau_1) \otimes FY(\tau_3), I\right)$.

Definition 7.2. For given labellings X and ι, define as their **amplitude** the evaluation of the labelled link diagram:

$$[X, \iota] \coloneqq \prod_{\tau \in \Delta_2} d(X(\tau)) \prod_{s \in \Delta_4} ⬠(FX(\tau_i), \iota(t_i)) \tag{7.2.3}$$

Here, the t_i are the faces of s and the τ_i their faces in turn, in the appropriate order. Whenever the orientation of the D^3 of a tetrahedron t_i induced from the total ordering matches the face orientation from the 4-simplex, evaluate the ⬠-quantity with the morphism $\iota(t_i)$ and otherwise with its dual basis vector. Since every tetrahedron is the face of exactly two 4-simplices, for every morphism $\iota(t)$, its dual will appear exactly once in the labelling, so the sum in the following will indeed range over dual bases.

Note that since the 2-handles are labelled with $F\Omega_{\mathcal{C}}$, the ⬠-diagram must be labelled with $FX(\tau_i)$.

From the normalisation from the multiple 4-simplices (0-handles), the evaluation of a Kirby diagram K is:

$$\langle K(F) \rangle = d(\Omega_{\mathcal{C}})^{1-|\Delta_4|} \sum_{\substack{\text{labellings} \\ X,\iota}} [X, \iota] \tag{7.2.4}$$

This quantity has to be multiplied by the normalisation, which is:

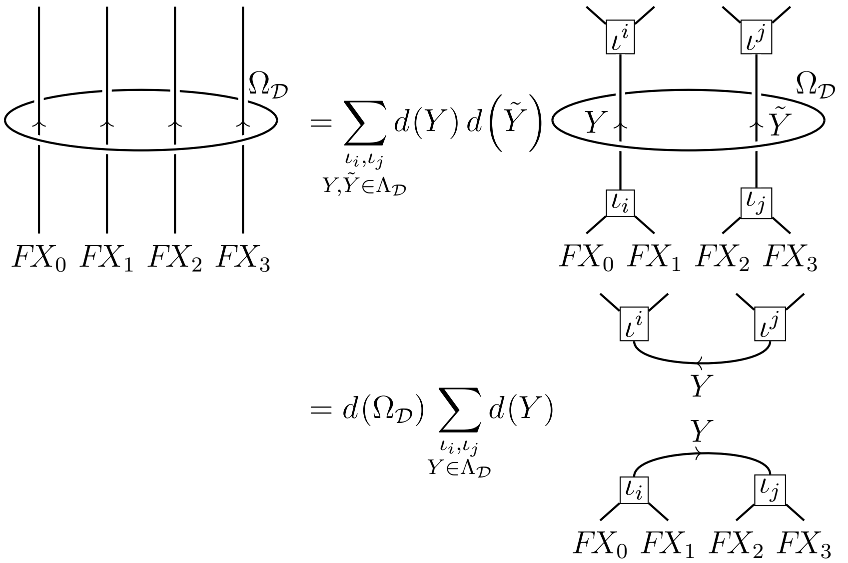
$$d(\Omega_{\mathcal{C}})^{-h_2+h_1} n^{-h_1} = \Omega_{\mathcal{C}}^{-|\Delta_2|+|\Delta_3|} d((F\Omega_{\mathcal{C}})')^{-|\Delta_3|}$$

Theorem 7.3. For $F\colon \mathcal{C} \to \mathcal{D}$ being a pivotal functor satisfying the conditions of Definition 5.5 with $\mathcal{D}$ modular, the generalised dichromatic invariant has the following state sum formula:

$$\begin{aligned} I_F(M) &= d(\Omega_{\mathcal{C}})^{1-|\Delta_2|+|\Delta_3|-|\Delta_4|} d((F\Omega_{\mathcal{C}})')^{-|\Delta_3|} \sum_{\substack{\text{labellings} \\ X,\iota}} [X, \iota] \\ &= d(\Omega_{\mathcal{C}})^{1-\chi(M)+|\Delta_0|-|\Delta_1|} d((F\Omega_{\mathcal{C}})')^{-|\Delta_3|} \\ &\quad \cdot \sum_{\substack{\text{labellings} \\ X,\iota}} \prod_{\tau \in \Delta_2} d(X(\tau)) \prod_{s \in \Delta_4} ⬠(FX(\tau_i), \iota(t_i)) \end{aligned} \tag{7.2.5}$$

7.3 Trading four-valent for trivalent morphisms

In order to compare it to the Crane-Yetter model, the state sum needs to be reformulated slightly. There, the vertices in the ⬠-diagram are trivalent, which is more convenient when working with $U_q sl(2)$ tilting modules. The four-valent morphisms appeared when applying Lemma 4.16 to the four 2-handles (triangles) going through a 1-handle (tetrahedron) in Proposition 6.13. If one inserts two $\Omega_{\mathcal{D}}$s instead, one can produce two trivalent vertices:

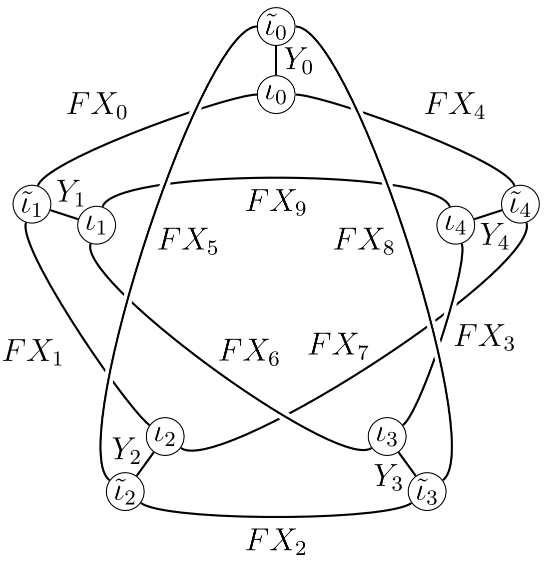


(7.3.1)

For the last step, Lemma 4.17 has been used, cancelling the factor $d(()\,\tilde{Y})$. Note that the additional objects now range over the simple objects in $\mathcal{D}$, not $\mathcal{C}$.

The alternative ⬠-quantity is then defined as:

$$\widetilde{\text{⬠}}\,(FX_i, Y_i, \iota_i, \tilde{\iota}_i) \coloneqq \text{⬠}\,(FX_0, \ldots, FX_9, Y_0, \ldots, Y_4, \iota_0, \ldots, \iota_4, \tilde{\iota}_0, \ldots, \tilde{\iota}_4)$$

$$:=$$ 

(7.3.2)

Again, it has to be specified where an object or its dual is the source or the target of a morphism. Each tetrahedron corresponds to an encirclement. It occurs as the face of two 4-simplices, which are oriented as submanifolds of M, and the tetrahedron inherits two opposite orientations from each of them. Orient the encircling (7.3.1) such that the 4-simplex from which the tetrahedron inherits the orientation agreeing with the ordering of the vertices appears on the top.

Object and morphism labellings now have different definitions than in Section 7.2:

Definition 7.4. An F-object labelling of the triangulation Δ is a pair of functions (X, Y), where

$$X\colon \Delta_2 \to \Lambda_{\mathcal{C}} \tag{7.3.3}$$

$$Y\colon \Delta_3 \to \Lambda_{\mathcal{D}} \tag{7.3.4}$$

Choose dual bases for the spaces $\mathcal{D}\,(FX(\tau_0) \otimes FX(\tau_2), Y(t))$ and $\mathcal{D}(FX(\tau_1) \otimes FX(\tau_3), Y(t))$ and their duals.

An F-morphism labelling is a pair of functions $(\iota, \tilde{\iota})$

$$\iota, \tilde{\iota}\colon \Delta_3 \to \operatorname{mor}\mathcal{D} \tag{7.3.5}$$

where $\iota(t)$ is a basis vector of the space $\mathcal{D}\,(FX(\tau_0) \otimes FX(\tau_2), Y(t))$ and $\tilde{\iota}(t)$ is a basis vector of $\mathcal{D}\,(FX(\tau_1) \otimes FX(\tau_3), Y(t))$.

Definition 7.5. For given labellings (X, Y) and $(\iota, \tilde{\iota})$, the amplitude is:

$$\begin{aligned}\langle (X,Y),(\iota,\tilde{\iota})\rangle \coloneqq \prod_{\tau\in\Delta_2} d(X(\tau)) \prod_{t\in\Delta_3} d(Y(t))\, d(\Omega_{\mathcal{D}}) \\ \cdot \prod_{s\in\Delta_4} \widetilde{\text{⛤}}(FX(\tau_i), Y(t_i), \iota(t_i), \tilde{\iota}(t_i))\end{aligned} \tag{7.3.6}$$

Lemma 7.6. From the Killing property and the normalisation from the multiple vertices, the evaluation of the special framed link L associated to the triangulation is:

$$\langle L\,(\Omega_{\mathcal{D}}, F\Omega_{\mathcal{C}})\rangle = d(\Omega_{\mathcal{C}})^{1-|\Delta_4|} \sum_{\substack{\text{labellings}\\ (X,Y),(\iota,\tilde{\iota})}} \langle (X,Y),(\iota,\tilde{\iota})\rangle \tag{7.3.7}$$

Theorem 7.7. Using the original Definition (5.2.1), the state sum formula can also be written as:

$$\begin{aligned} I_F(M) &= d(\Omega_{\mathcal{C}})^{1-|\Delta_2|+|\Delta_3|-|\Delta_4|}\, d(\Omega_{\mathcal{D}})^{-|\Delta_3|}\, d\big((F\Omega_{\mathcal{C}})'\big)^{-|\Delta_3|} \sum_{\substack{\text{labellings}\\ (X,Y),(\iota,\tilde{\iota})}} \langle (X,Y),(\iota,\tilde{\iota})\rangle \\ &= d(\Omega_{\mathcal{C}})^{1-\chi(M)+|\Delta_0|-|\Delta_1|}\, d\big((F\Omega_{\mathcal{C}})'\big)^{-|\Delta_3|} \\ &\quad \cdot \sum_{\substack{\text{labellings}\\ (X,Y),(\iota,\tilde{\iota})}} \Bigg(\prod_{\tau\in\Delta_2} d(X(\tau)) \prod_{t\in\Delta_3} d(Y(t)) \\ &\qquad\qquad \cdot \prod_{s\in\Delta_4} \widetilde{\text{⛤}}(FX(\tau_i), Y(t_i), \iota(t_i), \tilde{\iota}(t_i)) \Bigg) \end{aligned} \tag{7.3.8}$$

8 Examples

8.1 The Crane-Yetter state sum

If $F\colon \mathcal{C} \to \mathcal{D}$ is a full inclusion (Petit's dichromatic invariant, Example 5.15) and $\mathcal{D}$ is already modular, the generalised dichromatic invariant simplifies:

Proposition 8.1. Let $F\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be a full pivotal inclusion of a spherical fusion category into a modular category.

1. I_F depends only on $\mathcal{C}$, with the inherited ribbon structure. It will henceforth be denoted as $\widehat{CY}_{\mathcal{C}}$.

2. $\widehat{CY}_{\mathcal{C}}$ is the Crane-Yetter state sum $CY_{\mathcal{C}}$ from [CYK97] for $\mathcal{C}$ up to the Euler characteristic χ:

$$\widehat{CY}_{\mathcal{C}}(M) = CY_{\mathcal{C}}(M) \cdot d(\Omega_{\mathcal{C}})^{1-\chi(M)} \tag{8.1.1}$$

Proof. 1. Since $\mathcal{D}$ is modular, the simplified definition in Proposition 6.13 can be used, with $n = 1$. Object labellings already take values in $\Lambda_{\mathcal{C}}$. Morphism labellings take values in $\mathcal{D}(FX_1 \otimes \cdots \otimes FX_N, I)$, but this is isomorphic to $\mathcal{C}(X_1 \otimes \cdots \otimes X_N, I)$ since F is full. The evaluation of the Kirby diagram can thus be carried out in $\mathcal{C}$ and depends only on data from $\mathcal{C}$ and the ribbon structure inherited from $\mathcal{D}$.

2. In the state sum description, an additional $\Omega_{\mathcal{D}}$ is inserted in (7.3.1) to transform the four-valent vertex into two trivalent vertices, introducing additional objects X labelling the tetrahedra. Here, this can be achieved instead by using the insertion lemma 4.6 in $\mathcal{C}$. Thus the labellings of the state sum can be taken to range over $X\colon \Delta_3 \to \Lambda_{\mathcal{C}}$ and $\iota, \tilde{\iota}\colon \Delta_3 \to \operatorname{mor} \mathcal{C}$.

   A direct comparison of the state sum formula (7.3.8) to [CYK97, Theorem 3.2] shows the equality to $CY_{\mathcal{C}}$. The version of the insertion lemma 4.6 slightly differs by inserting $\Omega_{\mathcal{C}} = \bigoplus_X d(X)\, X$ whereas Crane, Yetter and Kauffman insert $\bigoplus_X X$, leading to different dimension factors.

□

Remark 8.2. Let $\mathcal{C}$ be a ribbon fusion category with braiding c. Then there is a full inclusion of $\mathcal{C}$ into its Drinfeld centre $\mathcal{Z}(\mathcal{C})$ by mapping an object X to $(X, c_{X,-})$. So the Crane-Yetter invariant can always be studied as a special case of Petit's dichromatic invariant. This is a significant generalisation since the original derivation of the Crane-Yetter state sum from a handlebody picture required $\mathcal{C}$ to be modular, while the version presented here does not.

Remark 8.3. Recall that if $\mathcal{D}$ is not modular, but modularisable, then the associated state sum model via the modularisation H can be considered. But $H \circ F$ will not always be full and may thus fail to give rise to a case of Petit's dichromatic invariant. However, if both categories are unitary, Corollary 6.11 can be used to return to a full inclusion, but in other cases, a new state sum model might arise.

Manifold M	$\widehat{CY}_{\mathcal{C}}(M)$	$\chi(M)$	$\sigma(M)$
$\mathbb{CP}^2$	$\sum_{X\in\Lambda_{\mathcal{C}}} d(X)^2\,\theta_X \cdot d(\Omega_{\mathcal{C}})^{-1}$	3	1
$\overline{\mathbb{CP}}^2$	$\sum_{X\in\Lambda_{\mathcal{C}}} d(X)^2\,\theta_X^{-1} \cdot d(\Omega_{\mathcal{C}})^{-1}$	3	-1
$S^2 \times S^2$	$d(\Omega'_{\mathcal{C}}) \cdot d(\Omega_{\mathcal{C}})^{-1}$	4	0
$S^2 \tilde{\times} S^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$	$d(\Omega'_{\mathcal{C}}) \cdot d(\Omega_{\mathcal{C}})^{-1}$	4	0
S^4 (including exotic candidates)	1	2	0

Table 8.1: The Crane-Yetter invariant for several simply-connected manifolds.

Simply-connected manifolds

For simply-connected manifolds, the Crane-Yetter invariant reduces to known invariants of the ribbon fusion category. Recalling the results from Section 5.4, the value for $\mathbb{CP}^2$ is:

$$I\left(\mathbb{CP}^2\right) = \frac{\sum_{X\in\Lambda_{\mathcal{C}}} \operatorname{tr}\left(\theta_X\right)}{d(\Omega_{\mathcal{C}})}$$

Since X is simple, the morphism θ_X amounts for multiplying by a complex number, which will be denoted by the same symbol:

$$= \frac{\sum_{X\in\Lambda_{\mathcal{C}}} d(X)^2\left(\theta_X\right)}{d(\Omega_{\mathcal{C}})} \tag{8.1.2}$$

The result is also known as the “normalised Gauss sum” of the category $\mathcal{C}$.

As another basic example, the manifold $S^2 \times S^2$ has the Hopf link of two 0-framed 2-handles as Kirby diagram, and thus its invariant is:

$$\widehat{CY}_{\mathcal{C}}\left(S^2 \times S^2\right) = \frac{d(\Omega_{\mathcal{C}'})}{d(\Omega_{\mathcal{C}})} \tag{8.1.3}$$

The same value could be calculated from (5.4.1), but in this case, the direct calculation is more convenient.

An overview over the Crane-Yetter invariant of several simply-connected 4-manifolds is given in Table 8.1.

8.2 Non-simply-connected manifolds

If the four-manifold M is not simply-connected, then the observation in Lemma 5.10 (that on simply-connected manifolds, the invariant is not stronger than Euler characteristic and signature) is not applicable any more. And indeed, already the Crane-Yetter invariant is stronger than the Broda invariant on such manifolds, in that it depends at least on the fundamental group. This can be seen in the following examples, and also in the next subsection.

Consider the Crane-Yetter model of a ribbon fusion category $\mathcal{C}$ that is not modular. This is, up to Euler characteristic and a constant factor, the generalised dichromatic invariant $\widehat{CY}_{\mathcal{C}}$ for a full inclusion F of $\mathcal{C}$ into a modular category $\mathcal{D}$.

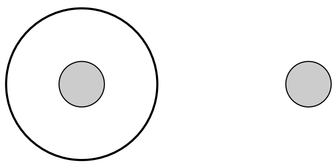
Manifolds of the form $S^1 \times M^3$

Assume for now that our manifold of interest is a product $S^1 \times M$, for some closed 3-manifold M. Since $S^1 \times M = \partial(D^2 \times M)$, its signature must be 0. The Euler characteristic is also $\chi(S^1 \times M) = \chi(S^1) \cdot \chi(M) = 0$.

Let us study the cases $M = S^3$ and $M = S^1 \times S^2$. The manifold $S^1 \times S^3$ has a handle decomposition with one 1-handle and no 2-handles and its link diagram in Akbulut notation is the dotted unknot. Its invariant is therefore:

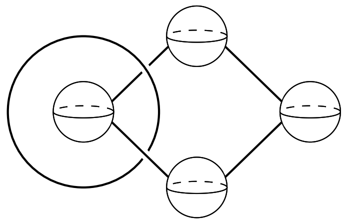
$$\widehat{CY}_{\mathcal{C}}\left(S^1 \times S^3\right) = d(\Omega_{\mathcal{C}}) \tag{8.2.1}$$

For $S^1 \times S^1 \times S^2$, a handle decomposition is derived by following [GS99, 4.3.1, 4.6.8 and 5.4.2], and starting from a Heegaard diagram of $S^1 \times S^2$. It is presented here in the form of a 2-handle attaching curve on the boundary of a solid torus, which is $\mathbb{R}^2 \cup \{\infty\}$ with two disks identified.

$S^1 \times S^2 =$ 

The two disks are the attaching disks of the 1-handle in $\partial D^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$. The circle is the attaching circle of the 2-handle. Thickening this picture gives a Kirby diagram for $I \times S^1 \times S^2$ and adding a further 1- and 2-handle gives:

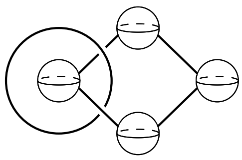
$$S^1 \times S^1 \times S^2 =$$



The left and the right 3-ball are the attaching balls of the thickened 1-handle, the front and the back ones come from the additional 1-handle.

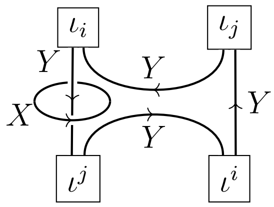
The simplified definition of the invariant from Proposition 6.13 is used. Since there are the same number of 2-handles and 1-handles and $n = 1$, the normalisation is 1, and the invariant evaluates to

$$\widehat{CY}_{\mathcal{C}}\left(S^1 \times S^1 \times S^2\right) = \Big\langle$$

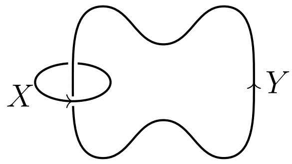


$$\Big\rangle$$

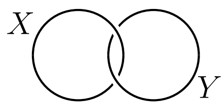
$$= \sum_{\substack{X,Y \in \Lambda_{\mathcal{C}} \\ \iota_i, \iota_j :\, Y^* \otimes Y \to I}} d(X)\, d(Y)$$



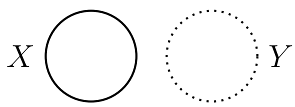
$$= \sum_{X,Y \in \Lambda_{\mathcal{C}}} d(X)\, d(Y)^{-1}$$



$$= \sum_{X,Y \in \Lambda_{\mathcal{C}}} d(X)\, d(Y)^{-1}$$



$$= \sum_{\substack{X \in \Lambda_{\mathcal{C}} \\ Y \in \Lambda_{\mathcal{C}'}}} d(X)\, d(Y)^{-1}$$



$$= \sum_{\substack{X \in \Lambda_{\mathcal{C}} \\ Y \in \Lambda_{\mathcal{C}'}}} d(X)^2$$

$$= |\Lambda_{\mathcal{C}'}|\, d(\Omega_{\mathcal{C}})\,. \qquad (8.2.2)$$

Manifold M	$\widehat{CY}_{\mathcal{C}}(M)$	$H^1(M)$	$\pi_1(M)$
$S^1 \times S^3$	$d(\Omega_{\mathcal{C}})$	$\mathbb{Z}$	$\mathbb{Z}$
$S^1 \times S^1 \times S^2$	$d(\Omega_{\mathcal{C}}) \cdot \lvert\Lambda_{\mathcal{C}'}\rvert$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$
$S^1 \times S^3 \# S^1 \times S^3 \# S^2 \times S^2$	$d(\Omega_{\mathcal{C}}) \cdot d(\Omega_{\mathcal{C}'})$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$

Table 8.2: The Crane-Yetter invariant for three non-simply-connected manifolds with zero Euler characteristic and signature, compared to their first homologies and their fundamental group. The notation $\mathbb{Z} * \mathbb{Z}$ stands for the free group product of $\mathbb{Z}$ with itself, i.e. the free group on two generators.

If $\mathcal{C}$ is not modular, that is, if $\Lambda_{\mathcal{C}'}$ has more than one element, $I(S^1 \times S^1 \times S^2) \neq I(S^1 \times S^3)$.

Homology and homotopy

Since $\widehat{CY}$ is multiplicative under connected sum, one can easily calculate the invariant on a manifold as the following:

$$\widehat{CY}_{\mathcal{C}}\left(S^1 \times S^3 \# S^1 \times S^3 \# S^2 \times S^2\right) = d(\Omega_{\mathcal{C}})\, d(\Omega_{\mathcal{C}'}) \tag{8.2.3}$$

This example is of interest since the latter manifold has the same first homology and signature as $S^1 \times S^1 \times S^2$, but a different fundamental group. The Crane-Yetter invariant is sensitive to this difference exactly iff the symmetric centre $\mathcal{C}'$ contains a simple object of dimension greater than 1. This situation occurs when $\mathcal{C}'$ is equivalent to the representations of a noncommutative finite group. An overview is given in Table 8.2.

Refined Broda invariant

An example of the Crane-Yetter invariant is the refined Broda invariant described in Section 5.5, where $\mathcal{C}$ is the subcategory of integer spins in a suitable quotient category $\mathcal{D}$ of tilting modules of $U_q sl(2)$, at an appropriate root of unity. According to [Rob97], the invariant for any manifold of the form $S^1 \times M^3$, with our normalisation, is:

$$\widehat{CY}_{\mathcal{C}} = 2^{b_1 - 1} d(\Omega_{\mathcal{C}}) \tag{8.2.4}$$

b_1 is the first $\mathbb{Z}_2$-coefficient Betti number of the four-manifold. A good example occurs for the root $q = \mathrm{e}^{\mathrm{i}\pi/4}$ (level 2), when the simple objects are the half-integer spin representations $\Lambda_{\mathcal{D}} = \left\{0, \frac{1}{2}, 1\right\}$ and $\Lambda_{\mathcal{C}} = \{0, 1\}$. In this example, $\mathcal{C} = \mathcal{C}' \simeq \operatorname{Rep}(\mathbb{Z}_2)$ is symmetric monoidal. If one takes a different non-trivial root of unity, $\mathcal{C}$ will not be symmetric monoidal any more, but it still has exactly two transparent objects.

Note that our results differ from those reported in [CKY93], where the authors implicitly assumed that $\mathcal{C}$ is modular, which it isn't.

8.3 Dijkgraaf-Witten models

The purpose of this section is to show how Dijkgraaf-Witten models are a special case of the Crane-Yetter model, and therefore of Petit's dichromatic invariant. The construction uses the representations of a finite group. The same symbol is used for a representation and its underlying vector space. If ρ_1 and ρ_2 are representations, then the trivial braiding is the map $c_{\rho_1,\rho_2}(x \otimes y) = y \otimes x$.

Definition 8.4. Let $F\colon \operatorname{Rep}(G) \hookrightarrow \mathcal{D}$ be a full ribbon inclusion of the representations of a finite group G, with the trivial braiding and trivial twist, into a modular category. Then the invariant I_F is called the **Dijkgraaf-Witten invariant** associated to G.

Remark 8.5. This choice of name will be justified subsequently. Since F is full, I_F can be denoted as $\widehat{CY}_{\operatorname{Rep}(G)}$ and only depends on G, as argued in Section 8.1. A suitable modular category to embed $\operatorname{Rep}(G)$ is simply the Drinfeld centre. Further comments on Dijkgraaf-Witten invariants as Crane-Yetter or Walker-Wang TQFTs are found in Section 9.2.

Definition 8.6. The **regular representation** of a finite group G is denoted as $\mathbb{C}[G]$ and defined as follows: The underlying vector space is the free vector space over the set G. The action of G is defined on the generators by left multiplication.

It is known that $\mathbb{C}[G] \cong \Omega_{\operatorname{Rep}(G)} \cong \oplus_\rho \rho \otimes \mathbb{C}^{d(\rho)}$ where ρ ranges over the irreducible representations of G.

Definition 8.7. Every group element $g \in G$ gives rise to a natural transformation of the fibre functor, $\mu(g)_\rho\colon \rho \to \rho$, given by $\mu(g)_\rho(v) = gv$. In fact, μ is a homomorphism.

The following two lemmas are basic facts of finite group representation theory.

Lemma 8.8. For any representation ρ, there is a projection on the invariant subspace:

$$\operatorname{inv}_\rho := \sum_i \rho \xrightarrow{\iota_i} I \xrightarrow{\iota^i} \rho \tag{8.3.1}$$

$$= \frac{1}{|G|} \sum_g \mu(g)_\rho \tag{8.3.2}$$

The ι_i and ι^j range over bases with $\iota_i \circ \iota^j = \delta_{i,j} 1_I$.

Lemma 8.9. The categorical trace over left multiplication on the regular representation, $\mu(g)_{\mathbb{C}[G]}$, is proportional to the delta function:

$$\operatorname{tr}\left(\mu(g)_{\mathbb{C}[G]}\right) = |G|\delta(g) \tag{8.3.3}$$

Definition 8.10. For a finite group G, a **flat G-connection** on a topological space M is a homomorphism $\pi_1(M) \to G$.

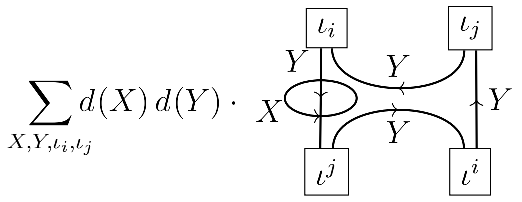
Remark 8.11. Only connections on four-manifolds will be considered here. Recall from Section 4.2 that the generators of the fundamental group $\pi_1(M)$ are given by the 1-handles, while each 2-handle is a relation word. Then a homomorphism $\pi_1(X) \to G$ is a choice of a group element for each 1-handle such that for every 2-handle, the group elements according to its relation word compose to the trivial element.

The following result shows that this invariant depends only on $\pi_1(M)$.

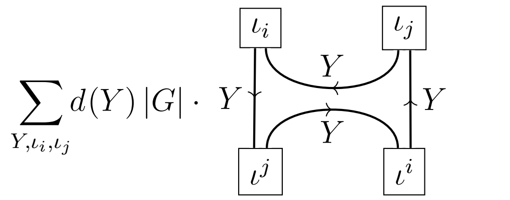
Theorem 8.12. *Let* $\operatorname{Rep}(G)$ *be the representations of a finite group G with the symmetric braiding and trivial twist. Then* $\widehat{CY}_{\operatorname{Rep}(G)}(M)$ *is the number of flat G-connections on M.*

Proof. The proof is graphical. Since $\mathcal{D}$ is modular, the simplified definition of the invariant from Proposition 6.13 can be used. Since F is full, the invariant can be calculated using objects and morphisms from $\mathcal{C}$, as in Proposition 8.1. The morphism $K(F)$ can be manipulated using the coherence axioms of ribbon categories as isotopies of the link in the plane. An example is given in Figure 8.1a, though one should bear in mind that in general there may be more than two 2-handle attaching curves passing along each 1-handle. There may also be crossings that cannot be removed by an isotopy alone.

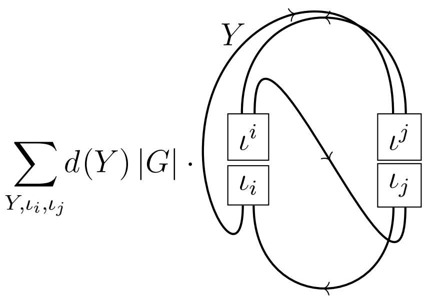
Consider any 2-handle in the link picture that is not linked to a 1-handle. Since $\operatorname{Rep}(G)$ is symmetric with trivial twist and F is ribbon, the knot on the 2-handle, its framing and links to other 2-handles can be undone, and then the morphism can be isotoped away. All such 2-handles then give a global numerical factor which cancels parts of the normalisation, arriving at a diagram that has only 2-handles which start



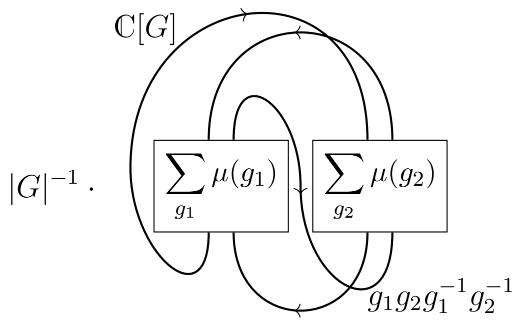
(a) Evaluation of a handle picture of a non-simply-connected manifold. (In this example, $S^1 \times S^1 \times S^2$).



(b) Remove the 2-handles not attached to any 1-handles to give a global factor.



(c) Rearrange the 1-handles to recognise the projection morphisms.



(d) 1-handles are generators of the fundamental group. Trace with the relation words.

Figure 8.1: For the representations of a finite group, $\widehat{CY}$ evaluates to the Dijkgraaf-Witten invariant.

or end in morphisms coming from 1-handles, while evaluating to the same value (Figure 8.1b).

The morphisms on the 1-handles are lined up horizontally and, after an isotopy, recognised as the projection morphisms $\mathrm{inv} = \frac{1}{|G|}\sum_g \mu(g)$ defined in Lemma 8.8. This is shown in Figures 8.1c and 8.1d. All of the 1-handles then give a morphism $\frac{1}{|G|^{h_1}} \sum_{g_1} \mu(g_1) \otimes \sum_{g_2} \mu(g_2) \otimes \cdots \otimes \sum_{g_{h_1}} \mu(g_{h_1})$, which are traced over with the 2-handles. The factor $\frac{1}{|G|^{h_1}}$ is cancelled by the normalisation as well since $\Omega_{\mathcal{C}} = |G|$.

To perform the trace for each 2-handle, consider Lemma 8.9. If the relation word for the 2-handle k is denoted by $r_1 r_2 \ldots r_{m_k}$, the trace for k is $\delta\left(g_{r_1} g_{r_2} \cdots g_{r_{m_k}}\right)$. Again the remaining normalisation is cancelled with the factor $|G|$. After tracing out with all 2-handles, the invariant is then

$$\begin{aligned}\widehat{CY}_{\mathrm{Rep}(G)}(M) &= \sum_{g_1\in G}\sum_{g_2\in G}\cdots\sum_{g_{h_1}\in G}\prod_{\text{2-handles } k}\delta\left(g_{r_1}g_{r_2}\cdots g_{r_{m_k}}\right)\\ &= \left|\{\phi\colon \pi_1(M)\to G\}\right| \end{aligned} \tag{8.3.4}$$

using Remark 8.11. □

This result shows that $\widehat{CY}_{\mathrm{Rep}(G)}$ is the partition function of a Dijkgraaf-Witten model, described for example in [Yet92]. In the more common normalisation in the literature, one would divide $\widehat{CY}_{\mathrm{Rep}(G)}$ by $|G| = d(\Omega_{\mathcal{C}})$, though.

Remark 8.13. One would expect a four-dimensional Dijkgraaf-Witten model to depend not only on a finite group G, but also on a 4-cocycle on G. The cocycle in the present model is trivial, though. A natural way for a 4-cocycle to arise is as a pentagonator in a tricategory. But braided categories are a special case of a tricategory with one 1-morphism, and these have a trivial pentagonator, see e.g. [CG07]. Hence, there seems little hope to introduce the data of a 4-cocycle into the representation category of G. The model would have to be generalised to fully weak monoidal bicategories, for example, following e.g. [Mac99].

Remark 8.14. Due to the Doplicher-Roberts reconstruction (see [Dri+10, Paragraph 2.12] for a categorical approach), it is known that symmetric fusion categories with trivial twist are essentially representation categories of finite supergroups. If the dimensions of all objects are required to be positive, the supergroup is in fact a group. So the case studied here is not much more restrictive than demanding that $\mathcal{C}$ be a symmetric fusion category.

8.4 Invariants from group homomorphisms

It is natural to consider generalising the Dijkgraaf-Witten examples by replacing the group G with a homomorphism $\phi\colon P \to G$. Any homomorphism can be factored into a surjective homomorphism followed by an inclusion, as $P \to \operatorname{Im}\phi \to G$. Taking the categories of unitary finite-dimensional representations leads to a functor

$$\phi^*\colon \mathrm{Rep}(G) \to \mathrm{Rep}(P)$$

given by composition with ϕ. It factors into functors $A\colon \mathrm{Rep}(G) \to \mathrm{Rep}(\operatorname{Im}\phi)$ followed by $B\colon \mathrm{Rep}(\operatorname{Im}\phi) \to \mathrm{Rep}(P)$. The first functor A is a restriction functor, which is a dominant functor. This follows from the fact that any $\operatorname{Im}\phi$-representation ρ is a subobject of $A(\operatorname{Ind}\rho)$, where Ind is the induction functor to P-representations. The second functor B is a full inclusion.

Trivial braiding

The first case to consider is when $\operatorname{Rep}(P)$ is augmented with the trivial braiding and trivial twist to make it a ribbon category, as in the Dijkgraaf-Witten invariant. Let $F\colon \operatorname{Rep}(P) \hookrightarrow \mathcal{D}$ be a full ribbon inclusion of $\operatorname{Rep}(P)$ with the trivial ribbon structure into a modular category.

Then the invariant $I_{F\circ\phi^*}$ generalises the Dijkgraaf-Witten invariant in principle but its evaluation is the same as a Dijkgraaf-Witten invariant. Indeed $F\circ\phi^* = F\circ B\circ A$. But A is dominant unitary and can be cancelled using Proposition 6.1, while $F\circ B$ is a full ribbon inclusion of $\operatorname{Rep}(\operatorname{Im}\phi)$ in $\mathcal{D}$, and so defines a Dijkgraaf-Witten invariant.

Despite the fact that the invariant is not new, the construction is still interesting because it may be a starting point for physical models. Just as in Proposition 8.1, the invariant can be calculated in the category $\operatorname{Rep}(P)$. The object labels are simple objects $X_i \in \operatorname{Rep}(G)$ and the morphism labels are a basis in $\operatorname{Rep}(P)\left(\phi^*X_1\otimes\ldots\otimes\phi^*X_N, I\right)$, or its dual space. The invariant is evaluated using the representation $p\mapsto \phi^*\mu_{\mathbb{C}[G]}(p) = \mu_{\mathbb{C}[G]}(\phi(p))$ with trace

$$\operatorname{tr}\mu_{\mathbb{C}[G]}\left(\phi(p)\right) = |G|\delta\left(\phi(p)\right)$$

using the delta-function in G. The projection morphisms are

$$\frac{1}{|P|}\sum_p \mu\left(\phi(p)\right).$$

Since the functor F is a full inclusion, the multiplicity n is just the multiplicity of I in $\phi^*\mathbb{C}[G]$. This can be calculated as $n = \frac{|G|}{|\operatorname{Im}\phi|}$. The formula for the invariant is thus

$$I_{F\circ\phi^*}(M) = \frac{1}{|\operatorname{Ker}\phi|^{h_1}}\sum_{p_1\in P}\sum_{p_2\in P}\cdots\sum_{p_{h_1}\in P}\prod_{\text{2-handles }k}\delta\left(\phi(p_{r_1}p_{r_2}\cdots p_{r_{m_k}})\right) \qquad (8.4.1)$$

Immediately, one can see that one can replace the δ-function in G by the one in $\operatorname{Im}\phi$ without changing the value of the invariant. Also each group element $\phi(p)$ appears exactly $|\operatorname{Ker}\phi|$ times, cancelling the normalisation. Thus one sees explicitly that the manifold invariant is the Dijkgraaf-Witten invariant of the subgroup $\operatorname{Im}\phi\subset G$.

Non-trivial braiding

A different construction from a group homomorphism is to consider cases where $\mathrm{Rep}(P)$ is augmented with a non-trivial braiding. Then one can consider the invariant I_{ϕ^*} directly, without needing the inclusion into a modular category. (Of course this also works with the trivial braiding, but then I_{ϕ^*} can be postcomposed with the fibre functor to vector spaces, Proposition 6.1 can be applied, and the invariant is equal to 1.)

Example 8.15. If $\phi\colon P \to G$ is injective, then $I_{\phi^*} = I_{1_{\mathrm{Rep}(P)}}$, which is a Broda invariant for the category $\mathrm{Rep}\, P$ and depends only on the Euler number and signature of the four-manifold.

Example 8.16. If $\phi\colon P \to G$ is surjective, then I_{ϕ^*} is a Petit dichromatic invariant.

Simple examples arise from $P = \mathbb{Z}_n$, the cyclic group of order n with the anyonic braiding [Maj00, Example 2.1.6] and the pivotal structure from Vect. The irreducible representations are one-dimensional and also labelled by $\mathbb{Z}_n$. The braiding on two irreducibles k, k' is

$$x \otimes y \mapsto e^{\frac{2\pi i}{n} kk'} y \otimes x$$

and so the transparent objects are $k = 0$, and also $k = n/2$ if n is even. In the case that n is odd, $\mathrm{Rep}(\mathbb{Z}_n)$ is modular and so the invariant of Example 8.16 only depends on $\mathrm{Rep}(G)$ with its induced ribbon structure. It is a Crane-Yetter invariant.

There are many more possible braidings [Dav97] and it seems an interesting project to explore the corresponding constructions of the invariant and Crane-Yetter models, which is left for future work.

9 Relations to TQFTs and physical models

This discussion section is written in a more informal style.

The invariants defined in this paper are related to various physical models. It is not just the value of the invariant that is important but also its construction in terms of data on simplices or handles. This is because in a physical model one is interested in features that are localised to lower-dimensional subsets, such as boundaries, corners or defects associated to embedded graphs, surfaces or other strata. In some cases it is possible to identify this data as the discrete version of a field in quantum field theory. In summary, the same invariant can extend to lower dimensions in different ways.

9.1 TQFTs from state sum models

Whenever there is a state sum formula for I_F, that is, when $\mathcal{D}$ is modular, it is possible to cast it in the form of a Topological Quantum Field Theory (TQFT) $\mathcal{Z}$, following a standard recipe [TV92].

- For a boundary manifold M^3 with a given triangulation Δ, define the set of labellings $L(M, \Delta)$ exactly like for the state sum model in Definition 7.1. Then define the free complex Hilbert space $Y(M, \Delta) := \mathbb{C}[L(M, \Delta)]$.
- For a cobordism $\Sigma^4 \colon M_1 \to M_2$ with triangulation Δ, the transition amplitude $\langle l_1| U(\Sigma, \Delta) |l_2\rangle$ is defined for the basis vectors coming from $l_{1,2} \in L(M_{1,2}, \Delta|_{1,2})$ via the state sum: Sum over all labellings of Σ that have l_1 and l_2 as boundary conditions. This gives a linear map $U(\Sigma, \Delta) \colon L(M_1, \Delta|_1) \to L(M_2, \Delta|_2)$. It is independent of the triangulation in the interior.
- $\mathcal{Z}$ assigns to an object M^3 the image of $U(I \times M)$. These spaces can be identified for different triangulations in a coherent way, again using cylinders. The resulting vector space is then independent of the triangulation of M.
- $\mathcal{Z}$ on morphisms Σ is defined by the restriction of U to the aforementioned spaces. Since a cylinder can always be glued to a cobordism without changing its isomorphism class, this is well-defined.

9.2 Walker-Wang models

By the previous subsection, Petit's dichromatic invariant I_F for a full inclusion $F \colon \mathcal{C} \hookrightarrow \mathcal{D}$ into a modular category extends to a Topological Quantum Field Theory $\mathcal{Z}$. More precisely, for a closed cobordism Σ^4,

$$\mathcal{Z}(\Sigma) = \frac{\widehat{CY}_{\mathcal{C}}(\Sigma)}{d(\Omega_{\mathcal{C}})^{1-\chi(\Sigma)}} \tag{9.2.1}$$

The denominator $d(\Omega_{\mathcal{C}})^{1-\chi(\Sigma)}$ is provided by comparison to the Crane-Yetter state sum (8.1.1).

It is believed that Walker-Wang TQFTs [WW12] are the Hamiltonian formulation of Crane-Yetter TQFTs.

This would imply that the dimensions of these state spaces for boundary manifolds M^3 can be calculated:

$$\begin{aligned}\dim \mathcal{Z}(M) &= \operatorname{tr} \mathbb{1}_{\mathcal{Z}(M)} \\ &= \mathcal{Z}\left(S^1 \times M\right) \\ &= \frac{I_{\mathcal{C}}\left(S^1 \times M\right)}{d(\Omega_{\mathcal{C}})} \end{aligned} \tag{9.2.2}$$

Non-trivial values of the invariant for manifolds of the form $S^1 \times M^3$ can then be interpreted as dimensions of state spaces of the corresponding TQFT. Comparing with Section 8.2 shows that these dimensions can indeed be greater than 1, as in the example of Broda's refined invariant.

As an example, for $M = S^1 \times S^2$, one arrives at $\dim \mathcal{Z}(S^1 \times S^2) = |\Lambda_{\mathcal{C}'}|$. This result is in excellent agreement with the analysis of Walker-Wang ground state degeneracies in [CBS13]. The state space of a TQFT corresponds to the space of ground states of the Hamiltonian.

If $\mathcal{C} \simeq \operatorname{Rep}(G)$ for G a finite group, the dimensions can be calculated explicitly, recalling Section 8.3:

$$\begin{aligned}\frac{\widehat{CY}_{\operatorname{Rep}(G)}\left(S^1 \times M\right)}{d(\Omega_{\mathcal{C}})} &= \frac{\left|\left\{\phi\colon \pi_1\left(S^1 \times M\right) \to G\right\}\right|}{|G|} \\ &= \frac{\left|\left\{\phi\colon \mathbb{Z} \times \pi_1(M) \to G\right\}\right|}{|G|} \\ &= \frac{\left|\left\{\left(\phi\colon \pi_1(M) \to G, g \in G\right) \middle| \phi = g\phi g^{-1}\right\}\right|}{|G|} \\ \text{(By Burnside's lemma)} \quad &= \left|\left\{\phi\colon \pi_1(M) \to G\right\} / \phi \sim g\phi g^{-1}\right| \end{aligned} \tag{9.2.3}$$

The state spaces are thus spanned by conjugacy classes of connections on the boundary manifolds, as one would expect if $\widehat{CY}_{\operatorname{Rep}(G)}$ extends as a Dijkgraaf-Witten TQFT.

9.3 Quantum gravity models

General relativity can be formulated in terms of connections and so it is natural to construct state sum models, or more generally quantum invariants of manifolds, that are modelled on connections. Usually the groups are Lie groups, but their representation categories are not fusion since the number of irreducibles is not finite. As a toy model therefore one can replace the Lie groups by finite groups to get an easy comparison with some of the invariants constructed above. A more sophisticated resolution of this problem is to use instead representations of quantum groups at a root of unity, which are indeed fusion categories. Finite groups are discussed here first and then some comments on the obstruction to using quantum groups in a similar way are made below.

Cartan connections can be thought of as principal G-connections that allow only gauge transformations of a subgroup $P \hookrightarrow G$. One of the motivations for the development of the generalised dichromatic invariant was the hope of arriving at a state sum model that could be interpreted as quantum Cartan geometry. Since there are formulations of general relativity in terms of Cartan geometry (see e.g. [Wis10]), this would give an interesting new approach to quantum gravity. However the constructions in Section 8.4 based on an inclusion $P \hookrightarrow G$ do not appear to lead to interesting new models.

A closely related construction is teleparallel gravity. This is based on a surjective homomorphism $P \to G$ with kernel N. According to Baez and Wise [BW12, theorem 32] the data for teleparallel gravity is a flat G-connection and a 1-form with values in the Lie algebra of N. For them, P is the Poincaré group and N the translation subgroup, but here the groups are allowed to be more general.

A flat G connection is easily described as an assignment of an element $g \in G$ to each 1-handle with a relation on each 2-handle, as in the Dijkgraaf-Witten model. The discrete analogue of the 1-form is the assignment of an element $n \in N$ to each 1-handle, with no relations on this data. For finite groups, this is exactly the data that is summed over in (8.4.1), the invariant associated to the homomorphism $\phi\colon P \to G$ that has kernel N. Two elements $p, p' \in P$ such that $\phi(p) = \phi(p')$ differ by an element $p^{-1}p' \in N$. This is the discrete analogue of the fact that the difference of two connection forms on a manifold is a 1-form. Thus the construction in (8.4.1) is a plausible finite group analogue of a sum over configurations of teleparallel gravity.

Quantum groups

Classical geometry works with Lie groups, which have an infinite number of irreducible representations. One hope would be to use quantum groups at a root of unity as a regularisation. However, few Lie group homomorphisms carry over to quantum groups. There are many examples of subgroups of Lie groups, but fewer sub-quantum groups of quantum groups are known. This is because most Lie group homomorphisms do not preserve the root system of the Lie algebras and thus neither the deformation.

And even for Hopf algebra homomorphisms, the restriction functor is not necessarily pivotal:

Example 9.1. As an example of a restriction functor that isn't pivotal, consider the category of tilting modules of $U_q sl(2)$ at an n-th root of unity. Its simple objects are spins $j \in \{0, \frac{1}{2}, \dots\}$. Recall that $\mathbb{C}[\mathbb{Z}_n]$ is a sub-Hopf algebra of $U_q sl(2)$. Recalling that $S^2 = SU(2)/U(1)$, one would hope that this Hopf algebra inclusion serves as Cartan geometry with a quantum 2-sphere.

The irreducible representations of $\mathbb{C}[\mathbb{Z}_n]$ are Fourier modes $\dots, -1, 0, 1, \dots$. Consider the restriction functor of representations, Res. It is obviously monoidal. Then $\mathrm{Res}\left(\frac{1}{2}\right) = -1 \oplus 1$. Both summands are invertible and thus have dimensions 1, whereas the quantum dimension of $\frac{1}{2}$ is generally not even an integer. Thus Res does not preserve quantum dimensions and can't be pivotal.

The crucial problem here is that the inclusion does not map the spherical element of $\mathbb{C}[\mathbb{Z}_n]$, which is 1, onto the spherical element of $U_q sl(2)$. A quantum group homomorphism of spherical quantum groups that preserves the spherical elements always gives rise to a pivotal functor on the representation categories [BMS12, Example 8.5]. However, no such homomorphism that gives rise to an invariant that is not a combination of the previously studied cases is known to the authors.

Spin foam models

Spin foam models are state sum models for quantum gravity constructed using representations of a quantum group, originally the "spins" of $U_q sl(2)$, hence the name. Starting with a Crane-Yetter state sum, a popular strategy in spin foam models is to impose constraints on the labels on the triangles and tetrahedra to mimick approaches to gravity as a constrained BF-theory [Bae00]. The unconstrained theory corresponds to the Crane-Yetter state sum, and different quantisation strategies of the classical constraints lead to different constraints, like in the Barrett-Crane [BC98] or the EPRL-model [Eng+08]. However, in these models the constraints on objects and morphisms typically spoil the monoidal product and so are not examples of the constructions presented here. An interesting question is whether it is possible to construct spin foam models of the type considered here, for example a spin foam model for teleparallel gravity. Such a model would involve studying the question of whether there are interesting quantum group analogues of a surjective homomorphism of groups.

9.4 Nonunitary theories

There are two possibilities to arrive at a theory which might be more general than the Crane-Yetter model. The first is to drop the assumption of the target category being modularisable; however this is a mild assumption which only specialises from supergroups to groups. Alternatively, when dropping the assumption that the categories are unitary, Lemma 6.3 is not applicable any more. To the knowledge of the authors, it is not known whether for a dominant pivotal functor will always satisfy $F\Omega_{\mathcal{C}} = n \cdot \Omega_{\mathcal{D}}$, so a counterexample might lead to an invariant that can't be reduced to a Crane-Yetter model.

9.5 Extended TQFTs

It is a common assumption that the Crane-Yetter model for modular $\mathcal{C}$ is an invertible four-dimensional extended TQFT. According to the cobordism hypothesis, it should correspond to an invertible (and therefore fully dualisable) object in a 4-category. The 4-category in question has as objects braided monoidal categories, as 1-morphisms monoidal bimodule categories (with an isomorphism between left and right action compatible with the braiding), as 2-morphisms linear bimodule categories, and furthermore bimodule functors and natural transformations.

A ribbon fusion category $\mathcal{C}$ acting on itself as a mere fusion category $\mathcal{M}$ from left and right should be an example for a fully dualisable, potentially noninvertible object. The object is $\mathcal{C}$ itself, while its dualisation data on the 1-morphism level is the bimodule data of $\mathcal{M}$. Being a fusion category, $\mathcal{M}$ is a bimodule over itself, giving the 2-morphism level of dualisation. The higher levels of dualisation should correspond to finite semisimplicity.

As has been suggested recently [HPT15, Section 3.2], a good notion of monoidal bimodule over a braided category is a braided central functor from $\mathcal{C}$ to $\mathcal{M}$, i.e. a braided functor $F\colon \mathcal{C} \to \mathcal{Z}(\mathcal{M})$. One would expect that the extended TQFT corresponding to such a bimodule is an extension of our (properly normalised) invariant for F, whenever it is also pivotal. And indeed, the inclusion $\mathcal{C} \to \mathcal{Z}(\mathcal{C})$ yields the Crane-Yetter model for $\mathcal{C}$.

10 Outlook

The generalised dichromatic invariant is a very versatile invariant in that it contains many previously studied theories as special cases. Table 10.1 gives an overview which functors give rise to several special cases. The generalised dichromatic invariant

Model	Pivotal functor F	Discussion
$U_q sl(2)$-Crane-Yetter state sum, Broda invariant	$1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ for $\mathcal{C}$ the tilting modules (spins) of $U_q sl(2)$	Example 5.14
Refined Broda invariant with $q = e^{i\pi/4}$	Canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ for $\mathcal{C} \simeq \operatorname{Rep} \mathbb{Z}_2$ generated by spins $\{0, 1\}$ and $\mathcal{D}$ all spins $\left\{0, \frac{1}{2}, 1\right\}$	Sections 8.2, 5.5 and 9.2
Refined Broda invariant, Crane-Yetter model for integer spins	Canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ for $\mathcal{C}$ integer spins and $\mathcal{D}$ all spins	Sections 8.2 and 5.5
Dijkgraaf-Witten TQFT for a finite group G	Any full inclusion of $\operatorname{Rep}(G)$ into a modular category, e.g. canonical inclusion $\operatorname{Rep}(G) \hookrightarrow \mathcal{Z}(\operatorname{Rep}(G))$	Sections 8.3 and 9.2
General Crane-Yetter state sum, Walker-Wang TQFT for $\mathcal{C}$ any ribbon fusion category	Any full inclusion of $\mathcal{C}$ into a modular category, e.g. canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C})$	Sections 8.1 and 9.2
Petit's dichromatic invariant	Any full inclusion $F\colon \mathcal{C} \hookrightarrow \mathcal{D}$ for $\mathcal{C}$ and $\mathcal{D}$ ribbon fusion categories	Example 5.15
"Generalised dichromatic state sum models"	Any functor into a modular category	Section 7.2

Table 10.1: Overview of the known special cases of the generalised dichromatic invariant, up to a factor of the Euler characteristic.

is at least as strong as the Crane-Yetter invariant, which is stronger than Euler characteristic and signature, although it is not known how strong exactly. If the additional constraints that the pivotal functor is unitary and the target category is modularisable are imposed, the generalised dichromatic invariant is exactly as strong as CY. In this situation, an upper bound for the strength of the state sum formula is probably given in [Fre+05]: Unitary four-dimensional TQFTs cannot distinguish homotopy equivalent simply-connected manifolds, or in general, s-cobordant manifolds. It remains to be demonstrated whether it is possible to construct a stronger, nonunitary TQFT with the present framework.

It is indicated in the literature [WW12] that the Walker-Wang model – and therefore also CY – for an arbitrary ribbon fusion category should factor into CY of its modularisation and its symmetric centre. The former reduces to the signature and the latter has been shown here to depend only on the fundamental group in the case of the symmetric centre being just the representations of a finite group. With the present framework, the conjecture can be formulated precisely:

Conjecture 10.1. Let $\mathcal{C}$ be a modularisable ribbon fusion category with $\mathcal{C}'$ its symmetric centre and $\widetilde{\mathcal{C}}$ its modularisation. Then $\widehat{CY}_{\mathcal{C}} = \widehat{CY}_{\mathcal{C}'} \cdot \widehat{CY}_{\widetilde{\mathcal{C}}}$.

The case of supergroups has not been treated here, but one would not expect it to differ much, except possibly a sensitivity to spin structures in the same manner as in the refined Broda invariant (Section 5.5).

The question whether the general case of the framework presented here is stronger than the mentioned special cases still remains open. Either way, motivated from solid state physics and TQFTs it would still be interesting to study how defects behave in the new models.

Part III

Half-ribbon categories

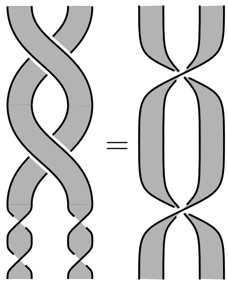


Figure 11.1: The balance equation relates the twist and the square of the braiding by a regular isotopy.

11 Introduction

Many braided monoidal categories carry a specific automorphism θ of the identity functor, the *twist*, which satisfies a compatibility equation with the braiding c, called the *balance equation*. It was introduced as (2.1.21) and will be recalled here:

$$\theta_{X\otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) \qquad (\mathrm{ax}_{c\theta})$$

This equation is most immediately understood in terms of its graphical representation, which is seen in Figure 11.1. The twist contains the complete information about the square of the braiding. It is natural to ask whether there is a "square root" of $(\mathrm{ax}_{c\theta})$, i.e. a square root of θ which contains the *full* information about braiding.

In the theory of Hopf algebras, such square roots are known for certain deformed universal enveloping algebras since [KR90] and [LS91] and have been studied in the context of ribbon Hopf algebras more recently [ST09]. The category theoretic viewpoint was left open at the time, though.

With another glance at Figure 11.1, it is convincing that a square root of the twist would have to be represented graphically by a half-twist[1], i.e. a turn of a ribbon (or several ribbons) by π. This ansatz seems to turn out in our favour: Figure 11.2 shows us how we would define the braiding in terms of a *half-twist* of two ribbons and half-twists of the individual ribbons. But a big question is glaring at us: What object in the category is represented by the back side of the ribbon? One possible answer is given by the graphical calculus of *involutive monoidal categories*, as defined in [Egg11], where half-twists in such categories are introduced as well. (Another answer

[1] Compare, for example, the following question asked on mathoverflow: http://mathoverflow.net/questions/28143/180°-vs-360°-twists-in-string-diagrams-for-ribbon-categories/204668

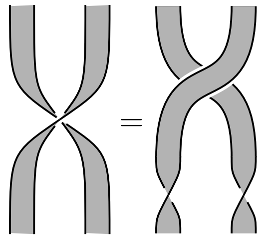


Figure 11.2: Turning two ribbons by π braids them.

are Selinger's "self-duality structures" [Sel10], which deserve separate treatment.)

The development of twists in braided monoidal categories with duals culminates in the notion of a "ribbon category", where the twist is compatible with the duals in the category. Such categories are ubiquitous in mathematical physics. However, the corresponding compatibility between *half-twists* and duals has only been addressed for the quantum group case in some terse remarks in [ST09]. This gap is filled here by *half-ribbon categories* in Definition 13.14.

Many other parts of the infrastructure of half-twists are presented here, such as half-twist preserving functors, strictification, compatibility with †-categories and a universal half-twist construction for balanced involutive categories.

The original motivation for the author to study half-twists was, maybe surprisingly, Noncommutative Geometry (NcG) in the sense of Alain Connes, or in particular, spectral triples [Con96]. A spectral triple consists of a unitary module of a $\star$-algebra, a real structure on the module, a Dirac operator and certain further structures and axioms which we will not detail here.

It is desirable to study spectral triples internal to a category, for example $\mathbb{Z}_2$-graded vector spaces or the representations of a quantum group, to study supersymmetric NcG or geometries with a quantum group symmetry, respectively. But what is a $\star$-algebra internal to a category? How do we abstract a real structure, and unitarity of the algebra action? The answer can be given in terms of involutive monoidal categories.

A half-twist is not immediately implied yet, but by studying the surprising correspondence between two-dimensional spin state sum models and spectral triples, a graphical calculus for NcG involving half-twists has been found by John Barrett (unpublished, in preparation). The half-twist represents the real structure on the module as well as the $\star$-operator on the algebra, and allows many axioms of spectral triples to correspond to isotopies of ribbons.

One has to require the spectral triple to be finite-dimensional (or, in categorical terms, dualisable), which is a smaller restriction than one might think, as it includes many special cases such as the fuzzy sphere [Mad92].

11.1 Outline

In Section 12, involutive monoidal categories are introduced. For the first two subsections, we mainly reproduce material from [Egg11] and [BM09], with the exception of one new example, namely involutive structures stemming from balanced monoidal categories. Subsequently, †-categories enriched in involutive monoidal categories are defined. An equivalent definition was already found by Egger, but unfortunately not published. We also revise the involutive structures coming from pivotal †-categories. We then present the known graphical calculus of involutive monoidal categories and its compatibility with braidings. To conclude, the new notion of *involutive pivotal category* is introduced, which will be expanded upon in the following section.

Section 13 is devoted to half-twists. In the first subsection, the definition and the most important properties from [Egg11] are recalled initially. We end by introducing *half-twist preserving* functors. Then, the main new concept is defined: Half-ribbon categories, which are involutive pivotal categories with a half-twist that is compatible with duals in a suitable sense. Such categories are shown to be indeed ribbon categories and to enjoy an intuitive graphical calculus. After defining half-twists in †-categories, examples are presented. In the remaining two subsections, some general constructions are presented. Half-twists can be, in a sense, strictified, and the implications of this finding are discussed. Finally, a general construction of a half-twist from a balanced involutive category is shown.

Open questions and ongoing research are discussed.

In this part, extra care is taken to differentiate the contributions of different authors to the field, since a lot of the material has been published several times independently. Furthermore, the relevant prerequisites will be supplied throughout the sections, so introductory parts and new results will sometimes be interleaved.

All content (such as definitions and theorems) by other authors is clearly labelled at the beginning of a block, and often only adapted in order to cohere with the conventions chosen here. Everything unlabelled is original content or new connections between known concepts.

Note also that in this part, we will usually not assume the blackboard framing as before, and therefore render most diagrams in ribbons.

12 Involutive monoidal categories

12.1 Generalising involutive monoids

Involutive monoids are ubiquitous in mathematics [Egg11, Examples 3.3]. Typical examples are the diverse notions of $\star$-algebras, where the involution is an antilinear anti-automorphism. One may ask how involutive monoids arise as internal objects of a category, and what structure such a category must carry.

For mere monoids, the answer is well known [BD97, Section 2.2] as the *microcosm principle*: We must first categorify the notion of a monoid to arrive at monoidal categories, and then we can define what an internal monoid is.

For involutive monoids, one could be puzzled first, since in some examples like $\star$-algebras, the involution is not even a morphism itself (as for example an antilinear map in the category of vector spaces and linear maps). Yet there exists a microcosm principle for involutive monoids [Jac12]. They have to be (vertically) categorified to *involutive monoidal categories*, and the riddle will be solved.

It is possible to horizontally categorify involutive monoids to †-*categories*[2], as well. The total situation, together with the relevant data, is displayed in Figure 12.1, and will be explained in detail in this section. The only omission will be involutive bicategories: They are not spelled out explicitly in the literature, nor will they appear here, since they are a straightforward generalisation of involutive monoidal categories.

Definition 12.1 (Folklore)**.** Let $(M, - \cdot - \colon M \times M \to M, 1 \in M)$ be a monoid, i.e. M a set and $- \cdot -$ an associative operation with unit 1. An involution for M is a function $\star \colon M \to M$, usually superscripted, satisfying for any $m, n \in M$:

$$n^\star \cdot m^\star = (m \cdot n)^\star$$

[2] Ironically and confusingly, †-categories have been called "categories with involution" in the past [Bur70].

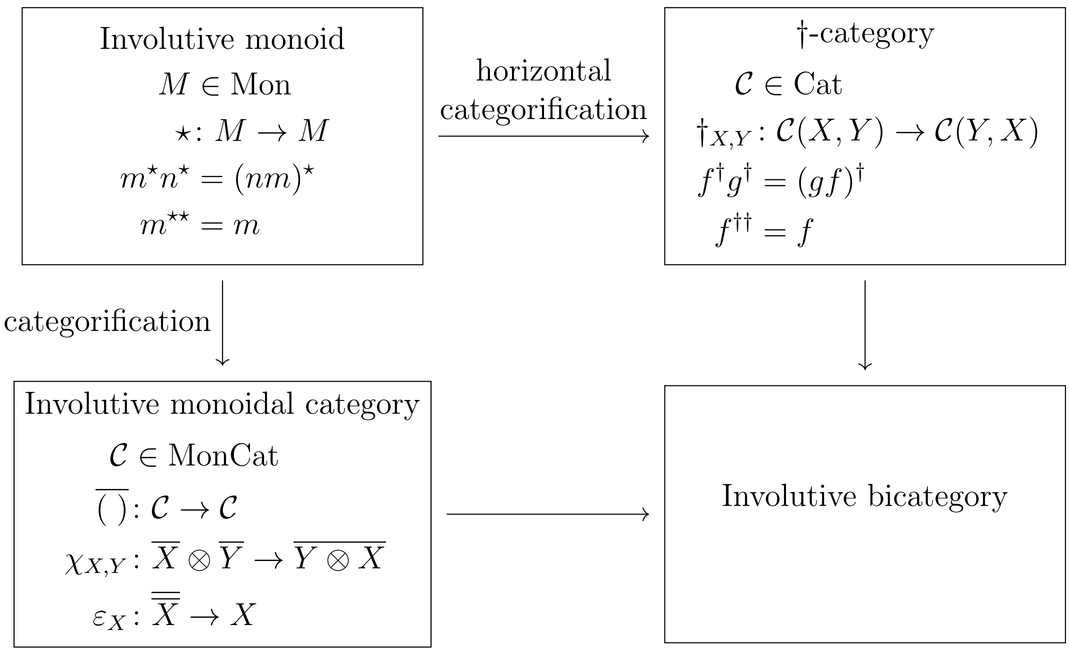


Figure 12.1: Horizontal and vertical categorification of involutive monoids

$$m^{\star\star} = m$$

Lemma 12.2 (Well-known)**.** The following is a sometimes forgotten lemma for involutive monoids:

$$1 = 1^{\star\star} = (1^\star \cdot 1)^\star = 1^\star \cdot 1^{\star\star} = 1^\star \cdot 1 = 1^\star$$

Again, it can be categorified in two ways: For involutive monoidal categories, it will yield a coherence isomorphism and an involutive monoid structure for the monoidal identity I, and for †-categories, it will yield a lemma for the identity morphisms of every object.

12.2 Involutive monoidal categories

Definition 12.3 ([Egg11, Definition 2.1])**.** Let $\mathcal{C} = (\mathcal{C}, -\otimes-, I, \alpha, \lambda, \rho)$ be a monoidal category. An **involutive structure** or **involution** for $\mathcal{C}$ consists of:

- A functor $\overline{(\,)}\colon \mathcal{C} \to \mathcal{C}$,
- a natural isomorphism $\chi_{X,Y}\colon \overline{X} \otimes \overline{Y} \to \overline{Y \otimes X}$,

- a natural isomorphism $\varepsilon_X \colon \overline{\overline{X}} \to X$,

satisfying the commutativity of the following diagrams:

$$\begin{array}{ccc} \big(\overline{X}\otimes\overline{Y}\big)\otimes\overline{Z} & \xrightarrow{\alpha_{X,Y,Z}} & \overline{X}\otimes\big(\overline{Y}\otimes\overline{Z}\big) \\ {\scriptstyle \chi_{X,Y}\otimes 1_{\overline{Z}}}\downarrow & & \downarrow{\scriptstyle 1_{\overline{X}}\otimes\chi_{Y,Z}} \\ \overline{Y\otimes X}\otimes\overline{Z} & & \overline{X}\otimes\overline{Z\otimes Y} \\ {\scriptstyle \chi_{Y\otimes X,Z}}\downarrow & & \downarrow{\scriptstyle \chi_{X,Z\otimes Y}} \\ \overline{Z\otimes(Y\otimes X)} & \xleftarrow[\overline{\alpha_{Z,Y,X}}]{} & \overline{(Z\otimes Y)\otimes X} \end{array} \qquad (F_3)$$

$$\begin{array}{ccc} \overline{\overline{X}}\otimes\overline{\overline{Y}} & \xrightarrow{\varepsilon_X\otimes\varepsilon_Y} & X\otimes Y \\ {\scriptstyle \chi_{\overline{X},\overline{Y}}}\downarrow & & \Big\Vert \\ \overline{\overline{Y}\otimes\overline{X}} & & \\ {\scriptstyle \overline{\chi_{X,Y}}}\downarrow & & \Big\Vert \\ \overline{\overline{X\otimes Y}} & \xrightarrow[\varepsilon_{X\otimes Y}]{} & X\otimes Y \end{array} \qquad (N_2)$$

$$\overline{\overline{\overline{X}}} \;\underset{\overline{\varepsilon_X}}{\overset{\varepsilon_{\overline{X}}}{\rightrightarrows}}\; \overline{X} \qquad (A)$$

A monoidal category with an involutive structure is an **involutive monoidal category**.

Remark 12.4. From here, we will usually suppress the familiar monoidal coherences α, ρ and λ, while explicitly writing out the involutive coherences χ or ε, even though it will turn out that they can be strictified.

The same notation will be used in all categories occurring, no confusion will arise.

Remark 12.5. It may come as a surprise that the involution functor is *covariant*, and not contravariant. But the wealth of natural examples for involutive monoidal categories suggest that the present definition is a good one.

A supposed contravariant involution functor would behave very similar to a dual functor with a pivotal structure, which can be related to an involutive structure only by means of an additional †-structure. This is discussed in Section 12.3.

Remark 12.6 ([Egg11, Remark 2.5]). The definition of involutive monoidal category is equivalent to that of a "strong bar category", as defined by Beggs and Majid in [BM09]. The term "involutive monoidal category" is preferred here, since it refers to the algebraic content of the definition, while "bar category" merely reflects its syntax.

Lemma 12.7 ([Egg11, Lemma 2.3]). Lemma 12.2 can be categorified to give the following canonical morphism $\mathring{\chi}$:

$$I \xrightarrow{\varepsilon_I^{-1}} \overline{\overline{I}} \xrightarrow{\overline{\rho_{\overline{I}}}^{-1}} \overline{\overline{I} \otimes I} \xrightarrow{\chi_{I,\overline{I}}^{-1}} \overline{I} \otimes \overline{\overline{I}} \xrightarrow{1_{\overline{I}} \otimes \varepsilon_I} \overline{I} \otimes I \xrightarrow{\rho_{\overline{I}}} \overline{I} \tag{def$_{\mathring{\chi}}$}$$

For more properties of this morphism, consult [Egg11, Lemma 2.3].

Remark 12.8 (Ehud Meir, in a private conversation). If we disregard the monoidal structure and χ, then $\overline{(\ )}$ and ε constitute an involutive category, as studied e.g. in [Jac12, Definition 2.1] and the references therein. Such a structure is exactly the same as a $\mathbb{Z}_2$-action.

Examples

Example 12.9 ([Egg11, Section 1]). Let $\mathcal{C}$ be a symmetric monoidal category with braiding c. Then $(1_{\mathcal{C}}, c, 1_{1_{\mathcal{C}}})$ is an involutive structure.

Example 12.10 ([Egg11, Example 2.2]). Let k be a ring with involution $\star$, for example $\mathbb{C}$ or $\mathbb{H}$. Then the monoidal category of bimodules over k has a natural involutive structure. For a bimodule M with left and right actions $- \triangleright -$ and $- \triangleleft -$, we choose for $\overline{M}$ the same underlying abelian group, and annotate its elements $\overline{m}, \overline{n}, \ldots$ with an overline. The left action of $\overline{M}$ is defined to be the right action of M, precomposed with the involution $\star$, and vice versa for the right action of $\overline{M}$:

$$a \triangleright \overline{m} := m \triangleleft a^\star \tag{12.2.1}$$

$$\overline{m} \triangleleft a := a^\star \triangleright m \tag{12.2.2}$$

The functor $\overline{(\ )}$ acts trivially on morphisms. The coherence isomorphism ε is simply the identity, and $\chi(m \otimes n) = n \otimes m$. It is easy to see that $\mathring{\chi} = \star$, where k is regarded as a bimodule over itself.

Proof. The newly defined left action is associative again:

$$a \triangleright b \triangleright \overline{m} = (m \triangleleft b^\star) \triangleleft a^\star = m \triangleleft (b^\star a^\star) = m \triangleleft (ab)^\star = ab \triangleright \overline{m} \tag{12.2.3}$$

All further axioms have analogous proofs and are left as exercises. □

Example 12.11 ([Egg11, Example 2.2]). Complex vector spaces, $\mathrm{Vect}_{\mathbb{C}}$, are an important special case of the previous example. For a complex vector space V, we define $\overline{V}$ as the same underlying abelian group, but the scalar multiplication differs. With μ_V the multiplication of V and $\mu_{\overline{V}}$ the multiplication of $\overline{V}$, we set $\mu_{\overline{V}}(\lambda, v) := \mu_V\left(\overline{\lambda}, v\right)$, where $v \in V, \lambda \in \mathbb{C}$ and $\overline{\lambda}$ is the complex conjugate.

The coherences are $\varepsilon(v) = v$, $\chi(v \otimes w) = w \otimes v$ and $\mathring{\chi}(\lambda) = \overline{\lambda}$.

Note that this is not the same involutive structure as the one arising from the symmetric structure.

Example 12.12 ([BM09, Example 2.3]). Let A be a complex $\star$-algebra. The monoidal category of A-bimodules has an involutive structure as well. It is defined as in 12.10, except that for the underlying vector space of $\overline{M}$, we choose the complex conjugate vector space of M.

Remark 12.13. The previous example can be generalised to bimodules of involutive monoids internal to any involutive monoidal category with sufficient colimits, although the details will not be carried out here.

Example 12.14 ([BM09, Proposition 3.1]). The category of Yetter-Drinfeld modules of a $\star$-Hopf algebra has an involutive structure. In the cited paper, more $\star$-Hopf algebra examples can be found.

Example 12.15. Let $\mathcal{C}$ be a balanced monoidal category, with braiding c and twist θ. Then $(1_{\mathcal{C}}, c, \theta)$ is an involutive structure.

Proof. (F_3) is the Yang-Baxter equation, rewritten with the braid axioms. (N_2) is the balance equation $(\mathrm{ax}_{c\theta})$ from page 77. Axiom (A) becomes a tautology. □

Involutive monoidal functors and natural transformations

Definition 12.16 ([Egg11, Definition 3.2]). An **involutive monoidal functor** between involutive monoidal categories consists of a strong monoidal functor (F, F^2, F^0)

and a natural isomorphism $\overline{F}_X \colon \overline{FX} \to F\overline{X}$ satisfying the following two axioms:

$$\begin{array}{ccccc}
\overline{FX} \otimes \overline{FY} & \xrightarrow{\overline{F}_X \otimes \overline{F}_Y} & F\overline{X} \otimes F\overline{Y} & \xrightarrow{F^2_{\overline{X},\overline{Y}}} & F\left(\overline{X} \otimes \overline{Y}\right) \\
{\scriptstyle \chi_{FX,FY}}\big\downarrow & & & & \big\downarrow{\scriptstyle F\chi_{X,Y}} \\
\overline{FY \otimes FX} & \xrightarrow{\overline{F^2_{Y,X}}} & \overline{F(Y \otimes X)} & \xrightarrow{\overline{F}_{Y \otimes X}} & F\left(\overline{Y \otimes X}\right)
\end{array} \tag{$\mathrm{ax}_{F\chi}$}$$

$$\begin{array}{ccc}
\overline{\overline{FX}} & \xrightarrow{\overline{\overline{F}_X}} & \overline{F\overline{X}} \\
{\scriptstyle \varepsilon_{FX}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{F}_{\overline{X}}} \\
FX & \xleftarrow[F\varepsilon_X]{} & F\overline{\overline{X}}
\end{array} \tag{$\mathrm{ax}_{F\varepsilon}$}$$

An **involutive natural transformation** $\beta \colon F \Rightarrow G$ is a monoidal natural transformations which is compatible with $\overline{F}$ and $\overline{G}$ in the following way:

$$\begin{array}{ccc}
\overline{FX} & \xrightarrow{\overline{\beta_X}} & \overline{GX} \\
{\scriptstyle \overline{F}_X}\big\downarrow & & \big\downarrow{\scriptstyle \overline{G}_X} \\
F\overline{X} & \xrightarrow{\beta_{\overline{X}}} & G\overline{X}
\end{array} \tag{12.2.4}$$

Theorem 12.1 ([Egg11, Corollary 3.6])**.** Every involutive monoidal category is involutively equivalent to a strict involutive monoidal category, i.e. one where the monoidal *and* involutive coherence morphisms are identities.

Internal involutive monoids

Definition 12.17 ([Egg11, Definition 5.5])**.** An **involutive object** in an involutive monoidal category is a pair $(X \in \mathrm{ob}\,\mathcal{C}, \tau \colon \overline{X} \xrightarrow{\cong} X)$ such that $\varepsilon_X \overline{\tau}\tau = 1_X$. The morphism τ is called **involution**.

Remark 12.18. Equivalently, an involution can be defined as a morphism from X to $\overline{X}$. (In an involutive category, $\mathcal{C}(X, \overline{X}) \cong \mathcal{C}(\overline{X}, X)$ canonically by application of $\overline{(\ )}$ and composition with ε.) The definition presented here follows the convention from [Egg11].

Example 12.19. In $\mathrm{Vect}_\mathbb{C}$, an involutive object is a complex vector space with a real structure, i.e. an antilinear involution.

Remark 12.20 ([Egg11, Remark 5.7]). Involutive objects are also called "$\star$-objects" [BM09, Definition 2.10], and "self-conjugates" [Jac12, Definition 3.1].

Definition 12.21 (Well-known, e.g. [BD95])**.** Let $(\mathcal{C}, - \otimes -, I)$ be a monoidal category. A **monoid internal to $\mathcal{C}$** consists of:

- An object M in $\mathcal{C}$,
- a morphism $- \cdot -\colon M \otimes M \to M$, the "multiplication",
- and a morphism $e\colon I \to M$, the unit.

The multiplication $- \cdot -$ is associative and has e as a right and left unit.

Definitions 12.22 ([Egg11, Example 3.3], [BM09, Definition 2.18, "star algebra"], [Jac12, Definition 5.1, "reversing"])**.** Let $(\mathcal{C}, - \otimes -, I, \overline{(\ )}, \chi, \epsilon)$ be an involutive monoidal category.

- Let $(M, - \cdot -, e)$ be a monoid internal to the underlying monoidal category. The **opposite monoid** is defined as $\left(\overline{M}, \overline{(- \cdot -)} \circ \chi, \overline{e} \circ \mathring{\chi}\right)$, and abbreviated as $\overline{M}$. It is a routine exercise to see that $\overline{M}$ is a monoid.
- An **involutive monoid internal to $\mathcal{C}$** is a monoid $(M, - \cdot -, e)$ internal to $\mathcal{C}$ which is also an involutive object in a compatible way, i.e. it carries a monoid homomorphism $\star\colon M \to \overline{M}$, such that $\varepsilon_M \overline{\star}\star = 1_M$.

Examples 12.23. An ordinary involutive monoid is an internal involutive monoid in Set, with the standard involutive structure from Example 12.9.

Real $\star$-algebras are involutive monoids internal to real vector spaces. Complex $\star$-algebras are involutive monoids internal to complex vector spaces, with complex conjugation as involutive structure.

12.3 †-categories

Definition 12.24 ([Sel07, Definition 2.2])**.** A **†-category** is a category $\mathcal{C}$ and, for each two objects X, Y, a map:

$$\dagger_{X,Y}\colon \mathcal{C}(X, Y) \to \mathcal{C}(Y, X)$$

We will drop the subscripts X and Y and denote it superscripted:

$$f^\dagger \coloneqq \dagger_{X,Y}(f)$$

The $\dagger$-structure has to satisfy these axioms:

$$g^\dagger f^\dagger = (fg)^\dagger$$
$$f^{\dagger\dagger} = f$$

A **$\dagger$-functor** is an ordinary functor F satisfying $F(f^\dagger) = (Ff)^\dagger$.

Remark 12.25 (Folk wisdom). A common, equivalent, definition of $\dagger$-categories is that of an identity-on-objects contravariant endofunctor that squares to the identity. The requirement of $\dagger$ as a functor being the identity on objects is often regarded as "evil", since sameness of objects is not preserved by isomorphism.

The definition presented here looks at $\dagger$-structures in a different way, namely as a categorification of an involution on a monoid. The requirement of $f^\dagger$ having source and target interchanged with respect to f can then be seen as a *typing rule*, very similar e.g. to the requirement that a composed morphism $f \circ g$ has exactly the same domain as g.

Lemma 12.26 (Well-known)**.** In a $\dagger$-category, we can horizontally categorify Lemma 12.2 for every object X:

$$1_X = 1_X^\dagger$$

Enriching $\dagger$-categories

$\dagger$-categories are horizontal categorifications of involutive monoids, and involutive monoids arise as the endomorphisms of an object in a $\dagger$-category (in the simplest case as a $\dagger$-category with one object). We have seen how to define involutive monoids internal to involutive monoidal categories, so it is natural to ask whether internal involutive monoids can be seen as endomorphism objects in enriched $\dagger$-categories. Basic knowledge of enriched categories is assumed for this part.

Definition 12.27. Let $\mathcal{V}$ an involutive monoidal category. A **$\dagger$-category enriched in $\mathcal{V}$** is a category $\mathcal{C}$ enriched in $\mathcal{V}$, and for every two objects $X, Y \in \operatorname{ob}\mathcal{C}$ an isomorphism $\dagger_{X,Y} \colon \mathcal{C}(X,Y) \to \overline{\mathcal{C}(Y,X)}$ in $\mathcal{V}$ making the following diagrams commute:

$$\begin{tikzcd} \mathcal{C}(X,Y) \arrow[r, "\dagger_{X,Y}"] \arrow[rr, bend right, "\varepsilon_{\mathcal{C}(X,Y)}"'] & \overline{\mathcal{C}(Y,X)} \arrow[r, "\overline{\dagger_{Y,X}}"] & \overline{\overline{\mathcal{C}(X,Y)}} \end{tikzcd} \qquad (\mathrm{ax}^{\mathrm{enr}}_{\dagger\dagger})$$

$$\begin{array}{ccc} \mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y) & \xrightarrow{\dagger_{Y,Z}\otimes\dagger_{X,Y}} & \overline{\mathcal{C}(Z,Y)}\otimes\overline{\mathcal{C}(Y,X)} \\ & & \downarrow{\scriptstyle \chi_{\mathcal{C}(Z,Y),\mathcal{C}(Y,X)}} \\ \downarrow{\scriptstyle -\circ-} & & \overline{\mathcal{C}(Y,X)\otimes\mathcal{C}(Z,Y)} \\ & & \downarrow{\scriptstyle \overline{-\circ-}} \\ \mathcal{C}(X,Z) & \xrightarrow{\dagger_{X,Z}} & \overline{\mathcal{C}(Z,X)} \end{array} \qquad (\mathrm{ax}^{\mathrm{enr}}_{\circ\dagger})$$

An **enriched †-functor** is an ordinary enriched functor F satisfying:

$$\begin{array}{ccc} \mathcal{C}(X,Y) & \xrightarrow{\dagger_{X,Y}} & \overline{\mathcal{C}(Y,X)} \\ {\scriptstyle F_{X,Y}}\downarrow & & \downarrow{\scriptstyle \overline{F_{Y,X}}} \\ \mathcal{D}(FX,FY) & \xrightarrow{\dagger_{FX,FY}} & \overline{\mathcal{D}(FY,FX)} \end{array} \qquad (12.3.1)$$

$F_{X,Y}$ is the action of the functor on the hom-object.

Remark 12.28. An equivalent definition has been given by Egger at a talk in Oxford in 2011, unfortunately unpublished. The present definition was derived by the author independently and is given for the sake of completeness.

Remark 12.29. It is an easy exercise to see that †-categories enriched in Set (with the involutive structure coming from the symmetric structure) are just ordinary (locally small) †-categories.

We can also recover from a $\mathcal{V}$-enriched †-category $\mathcal{C}$ an ordinary †-category $\mathcal{C}_0$ with the same objects by defining the morphism sets in the following standard way:

$$\mathcal{C}_0(X,Y) = \mathcal{V}(I,\mathcal{C}(X,Y)) \qquad (12.3.2)$$

The †-structure $\dagger_0$ is defined by the following commutative diagram in $\mathcal{V}$:

$$\begin{array}{ccc} I & \xrightarrow{f} & \mathcal{C}(X,Y) \\ {\scriptstyle \mathring{\chi}}\downarrow & & \downarrow{\scriptstyle \dagger_{X,Y}} \\ \overline{I} & \xrightarrow{\overline{f^{\dagger_0}}} & \overline{\mathcal{C}(Y,X)} \end{array} \qquad (\mathrm{def}_{\dagger_0})$$

Remark 12.30. The viewpoint of the underlying ordinary †-category is relevant to define essential properties of individual morphisms usually available in †-categories, such as *unitarity*, *self-adjointness* and *positivity*. For example, a morphism f in an ordinary †-category is unitary if it satisfies $f^{-1} = f^{\dagger}$. An enriched morphism $f\colon I \to \mathcal{C}(X,Y)$ is unitary if the following diagrams in $\mathcal{V}$ commute:

$$\begin{tikzcd} & \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,X) \arrow[dr, "-\circ-"] & \\ I \arrow[ur, "f\otimes f^{\dagger_0}"] \arrow[rr, "1_X"] \arrow[dr, "f^{\dagger_0}\otimes f"'] & & \mathcal{C}(X,X) \\ & \mathcal{C}(Y,X)\otimes\mathcal{C}(X,Y) \arrow[ur, "-\circ-"'] & \end{tikzcd}$$

Such a definition is important when defining enriched †-categories with additional structures, such as monoidal, braided, or balanced structures, since these are usually required to be unitary. Whether it is possible to express unitarity internally, for example as a subobject $\mathcal{C}_u(X,Y) \hookrightarrow \mathcal{C}(X,Y)$ of unitary morphisms, is a different matter and will not be explored here.

Examples 12.31. Enriched †-categories are abundant in functional analysis and quantum algebra.

- A complex $\star$-category is a †-category enriched in complex vector spaces with complex conjugation as involutive structure.
- A C^*-category is a †-category enriched in complex Banach spaces with the projective tensor product, and complex conjugation as involutive structure satisfying certain axioms, see e.g. [GLR85].

Definition 12.32. Let $\mathcal{C}$ be a †-category enriched in an involutive monoidal category $\mathcal{M}$. For any object $X \in \mathrm{ob}\,\mathcal{C}$, the endomorphism object $\mathcal{C}(X,X) \in \mathrm{ob}\,\mathcal{M}$ forms an involutive monoid, with the identity $1_X\colon I \to \mathcal{C}(X,X)$ as unit, composition $-\circ-\colon \mathcal{C}(X,X)\otimes\mathcal{C}(X,X) \to \mathcal{C}(X,X)$ as multiplication and $\dagger_{X,X}\colon \mathcal{C}(X,X) \to \overline{\mathcal{C}(X,X)}$ as involution.

Proof. Associativity and unit laws follow directly from the axioms of enrichment. $\dagger_{X,X}$ is a monoid homomorphism as a consequence of Axiom ($\mathrm{ax}^{\mathrm{enr}}_{\circ\dagger}$) from page 88. □

Example 12.33. A $\star$-category is a †-category enriched in $\mathrm{Vect}_{\mathbb{C}}$ with complex conjugation as involutive structure. The †-operation is therefore antilinear.

While the following definition is not new, its formulation in terms of enriched †-categories and enriched †-functors is.

Example 12.34. Let A be a $\star$-algebra, and BA its delooping as a $\star$-category. Furthermore, let $\mathcal{C}$ be a $\star$-category. The **category of unitary** A**-modules** in $\mathcal{C}$ is the category of linear †-functors $\rho\colon BA \to \mathcal{C}$. It inherits the complex linear structure and the †-structure from $\mathcal{C}$ pointwise.

Pivotal †-categories

It is maybe surprising that in an involutive monoidal category, the involution functor is covariant. A more well known categorification of involutions are duals. If a category is required to be rigid, we can arbitrarily choose a right and left dual for each object. This defines right and left dual functors $(\ )^*$ and ${}^*(\ )$ and natural isomorphisms $(X \otimes Y)^* \cong Y^* \otimes X^*$. If the category is also pivotal, an isomorphism $i_X\colon X \to X^{**}$ exists, so the structure of duals looks suspiciously similar to an involutive monoidal structure, with the crucial difference that the dual functors are contravariant.

A †-structure, on the other hand, flips the direction of the morphisms, so it seems natural to combine the two to define an involutive monoidal structure.

Definition 12.35 ([Sel10, Section 7.3])**.** A **monoidal †-category** is a †-category that is also monoidal, such that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms, and the monoidal coherences are unitary.

A **pivotal †-category** is a monoidal †-category that is also rigid, such that $f^{*\dagger} = f^{\dagger *}$ and the canonical isomorphisms $(X \otimes Y)^* \cong Y^* \otimes X^*$ and $I \cong I^*$ are unitary.

Such a category has a natural unitary pivotal structure, which we choose.

Lemma 12.36 ([Egg11, Lemma 6.1])**.** A pivotal †-category has a canonical involutive monoidal structure. The involution functor is the covariant dual functor $\overline{(\ )} := (\ )_* := (\ )^{*\dagger}$, χ is given by the canonical isomorphisms $(X \otimes Y)^* \cong Y^* \otimes X^*$, and ε is the inverse of the pivotal structure.

Remark 12.37. "Involution monoids" from [Vic11, Definition 2.20] are involutive monoids in a pivotal †-category.

Example 12.38 ([Egg11, Section 6]). The category of finite dimensional Hilbert spaces, $\mathrm{Hilb}_{f.d.}$, is pivotal and has a †-structure, therefore it is involutive monoidal.

Example 12.39. In the category of oriented bordisms in any dimension, the dual of a boundary manifold is the same manifold with opposite orientation. The dual of a bordism is simply the same bordism, with source and target exchanged. Thus, a trivial pivotal structure can be given. The †-structure on a bordism is given by orientation reversal (and exchange of source and target).

Combining duals and †-structure yields a beautiful involutive monoidal structure: The involution functor is given by reversing the orientation of the bordism and the boundary simultaneously (while not exchanging source and target). ε is trivial, and χ simply interchanges two manifolds.

12.4 Graphical calculus and balanced categories

Monoidal categories have a graphical calculus where morphisms are drawn in the plane, with composition in the vertical direction and the monoidal product in the horizontal direction. It can be extended to involutive monoidal categories.

A good graphical calculus should assign a geometrical operation on the diagram to a functor, such that coherence isomorphisms are invisible, or at least invisible up to a suitable kind of isotopy. But what geometrical operation can we assign to the involution functor $\overline{(\,)}$ such that $\chi_{X,Y} \colon \overline{X} \otimes \overline{Y} \to \overline{Y \otimes X}$ is invisible?

Two possibilities spring to mind: Involution could be rotation around the *vertical* axis by π, or reflection, again at the vertical axis. The two choices don't differ for planar diagrams, but if we introduce braidings and twists, the distinction will matter, this is discussed in Remark 12.42.

The former, namely rotation around the vertical axis, will turn out to be the correct graphical representation as soon as we consider half-twists in Section 13.1. The back side of the ribbon will be represented by a darker shade, as can be seen in Figure 12.2.

Remark 12.40. The graphical representation of dualisation in a rigid category is rotation by π around the horizontal axis perpendicular to the plane, and the graphical representation of a †-structure is reflection at the horizontal axis in the plane, so the involution has a graphical interpretation which is different from the aforementioned ones.

Definition 12.41 ([BM09, Definition 4.1], [Egg11, Remark 4.6])**.** Let $\mathcal{C}$ be a balanced monoidal category with involutive structure $\left(\overline{(\,)}, \chi, \varepsilon\right)$. $\mathcal{C}$ is called a **balanced involutive monoidal category** if the braiding c and twist θ are compatible with

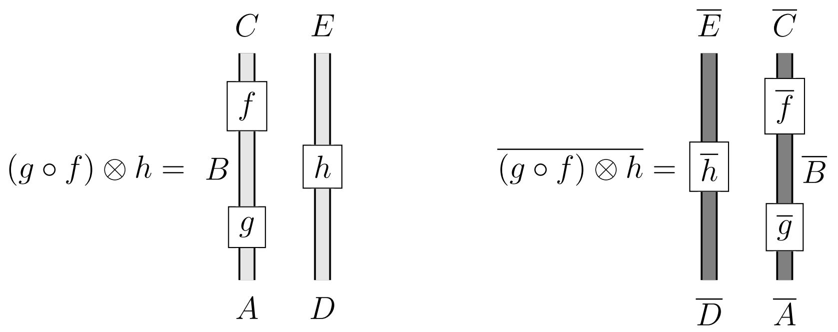


Figure 12.2: Graphical calculus of involutive monoidal categories, where the involution is represented by a rotation around the vertical axis by π. Recall that $\overline{A \otimes D} \cong \overline{D} \otimes \overline{A}$.

the involutive structure, in the sense that they satisfy the following two axioms for any objects X, Y:

$$\theta_{\overline{X}} = \overline{\theta_X} \qquad (\mathrm{ax}_{\overline{\theta}})$$

$$\begin{array}{ccc} \overline{X} \otimes \overline{Y} & \xrightarrow{c_{\overline{X},\overline{Y}}} & \overline{Y} \otimes \overline{X} \\ {\scriptstyle \chi_{X,Y}} \downarrow & & \downarrow {\scriptstyle \chi_{Y,X}} \\ \overline{Y \otimes X} & \xrightarrow{\overline{c_{Y,X}}} & \overline{X \otimes Y} \end{array} \qquad (\mathrm{ax}_{\overline{c}})$$

Remark 12.42 ([BM09, Section 4.1], [Egg11, Remark 4.6]). Balanced involutive structures were originally called “real” balanced structures or braidings. “Antireal” (or “hermitian”) balanced structures can be defined as well by replacing $\theta_{\overline{X}}$ and $c_{\overline{X},\overline{Y}}$ by their inverses in the axioms $(\mathrm{ax}_{\overline{\theta}})$ and $(\mathrm{ax}_{\overline{c}})$.

Remark 12.43 ([Egg11, Remark 4.6], private communication with P. Selinger). The two graphical representations of the involutive structure – rotation by π and reflection – correspond to balanced involutive (or real balanced) structures and antireal balanced structures, respectively. The axioms $(\mathrm{ax}_{\overline{c}})$ and $(\mathrm{ax}_{\overline{\theta}})$ verify that the two ways to interpret the diagram of the braiding (or twist) labelled by objects $\overline{X}$ and $\overline{Y}$ coincide.

Examples 12.44. Examples 12.9, 12.11 and 12.15 for involutive monoidal categories from Section 12.2 are also balanced involutive, as can be easily seen. As a notable exception, the category of Yetter-Drinfeld modules (Example 12.14) has an antireal balanced structure [BM09, Example 4.3].

Remark 12.45. From the viewpoint of the graphical calculus, Example 12.15 (constructing an involutive structure from a balanced structure) can easily mislead.

The coherence morphisms χ and ε are invisible graphically, but they are the same morphisms as braiding and twist, which of course have a visible representation. They can only be distinguished by the application of the involution functor to objects.

12.5 Involutive pivotal categories

Pivotal categories have a graphical calculus in which morphisms are rotated around an axis pointing out of the plane. Furthermore, pivotal structures relate left and right duals. Involutive structures rotate the diagram around a vertical axis, thus interchanging left and right. Therefore it is interesting to find out how involutive and pivotal structures can be compatible.

For chosen left and right duals ${}^*X$ and X^* in a rigid category, we will sometimes suppress the canonical natural isomorphisms $l_X : X \xrightarrow{\cong} ({}^*X)^*$ and $\tilde{l}_X : X \xrightarrow{\cong} {}^*(X^*)$. For example, we will write $i_{{}^*X} : {}^*X \to X^*$ for the pivotal isomorphism of a left dual.

Definition 12.46 (Well known)**.** Let $\mathcal{C}$ be a monoidal category. The monoidal category $\mathcal{C}^{\mathrm{rev}}$ has the same underlying category and the reverse monoidal product $X \otimes^{\mathrm{rev}} Y := Y \otimes X$.

Whenever $\mathcal{C}$ is rigid with chosen right and left duals X^* and ${}^*X$, we choose in $\mathcal{C}^{\mathrm{rev}}$ the duals $X^\vee := {}^*X$ and ${}^\vee X := X^*$.

Consequently, if $\mathcal{C}$ is equipped with a pivotal structure i, the monoidal reverse category $\mathcal{C}^{\mathrm{rev}}$ has a canonical pivotal structure ι, which is the inverse of i, up to the aforementioned isomorphisms $X^{**\vee\vee} = {}^{**}(X^{**}) \cong X$.

Remark 12.47 ([Egg11, Section 4])*.* An involutive structure on a monoidal category $\mathcal{C}$ gives rise to a monoidal functor $\overline{(\ )} : \mathcal{C} \to \mathcal{C}^{\mathrm{rev}}$, with coherences χ and $\mathring{\chi}$.

This is of course motivated by the graphical calculus of rotation by π (or reflection), which interchanges the order of objects in a monoidal product.

Lemma 12.48 (Compare [BM09, Proposition 6.2] and [ST09, Comment 4.12])**.** For left and right duals in an involutive monoidal category, the following canonical natural isomorphisms exist:

$$k_X : \overline{X^*} \xrightarrow{\cong} \overline{X}^\vee = {}^*\overline{X} \tag{12.5.1}$$

$$\widetilde{k}_X : \overline{{}^*X} \xrightarrow{\cong} {}^\vee\overline{X} = \overline{X}^* \tag{12.5.2}$$

Proof. $\overline{(\ )} : \mathcal{C} \to \mathcal{C}^{\mathrm{rev}}$ is monoidal and thus preserves duals up to canonical natural isomorphisms.

Explicitly: Let X have a right dual, X^*. The following diagram defines a left evaluation $\widetilde{\mathrm{ev}}$:

$$\begin{array}{ccc}
\overline{X^*} \otimes \overline{X} & \xrightarrow{\widetilde{\mathrm{ev}}} & I \\
{\scriptstyle \chi_{X^*,X}} \downarrow & & \downarrow {\scriptstyle \mathring{\chi}} \\
\overline{X \otimes X^*} & \xrightarrow{\overline{\mathrm{ev}_X}} & \overline{I}
\end{array}$$

It is easy to see that the corresponding definition of $\widetilde{\mathrm{coev}}$ satisfies the snake identities. Thus, $\overline{X^*}$ is a left dual for $\overline{X}$, and any chosen left dual ${}^*X$ is canonically isomorphic to it.

For $\widetilde{k}$, an analogous argument holds. $\square$

Corollary 12.49 ([BM09, Proposition 6.2])**.** An involutive monoidal category which is left rigid is also right rigid, and vice versa.

Proof. For any object X, we have $\overline{{}^*\overline{X}} \cong \overline{\overline{X}}^*$ as a right dual for $\overline{\overline{X}} \cong X$. $\square$

Definition 12.50. An **involutive pivotal category** $(\mathcal{C}, - \otimes -, I)$ is an involutive monoidal category with pivotal structure, such that the monoidal functor $\left(\overline{(\ )}, \chi, \mathring{\chi}\right) : (\mathcal{C}, - \otimes -, I) \to (\mathcal{C}, - \otimes^{\mathrm{rev}} -, I)$ is pivotal.

Remark 12.51. Again denoting the right dual of an object X in $\mathcal{C}^{\mathrm{rev}}$ by $X^\vee$, pivotality of $\overline{(\ )}$ amounts to the following commutative diagram:

$$\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{i_X}} & \overline{X^{**}} \\
{\scriptstyle \iota_{\overline{X}}} \downarrow & & \downarrow {\scriptstyle k_{X^*}} \\
\overline{X}^{\vee\vee} & \xrightarrow{k_X^\vee} & \overline{X^*}^\vee
\end{array} \tag{12.5.3}$$

In terms of left duals in $\mathcal{C}$, this amounts to:

$$\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{i_X}} & \overline{X^{**}} \\
{\scriptstyle l_{\overline{X}}} \downarrow & & \Big| \\
\left({}^*\overline{X}\right)^* & & \Big| \\
{\scriptstyle l^*_{{}^*\overline{X}}} \uparrow & & \Big| {\scriptstyle k_{X^*}} \\
\left({}^{**}\overline{X}\right)^{**} & & \Big| \\
{\scriptstyle i_{{}^{**}\overline{X}}} \uparrow & & \downarrow \\
{}^{**}\overline{X} & \xrightarrow{{}^*k_X} & {}^*\overline{X^*}
\end{array} \tag{piv}$$

Definition 12.52. In an involutive pivotal category, the following canonical isomorphisms exist:

$$j_X \colon \overline{X^*} \xrightarrow{\cong} \overline{X}^* \tag{def$_j$}$$

They are defined as:

$$\overline{X^*} \xrightarrow{k_X} {}^*\overline{X} \xrightarrow{i_{{}^*\overline{X}}} \left({}^*\overline{X}\right)^{**} \xrightarrow{l^*_{\overline{X}}} \overline{X}^*$$

Remark 12.53. Since $\overline{(\ )}$ is pivotal, this is the same as the following alternative definition:

$$\overline{X^*} \xrightarrow{\overline{\tilde{l}_{X^*}}} \overline{{}^*(X^{**})} \xrightarrow{\overline{{}^*i_X}} \overline{{}^*X} \xrightarrow{\tilde{k}_X} \overline{X}^*$$

More explicitly, for an object X, construct the following right evaluation ev for $\overline{X}$:

$$\begin{array}{ccc}
\overline{X} \otimes \overline{X^*} & \xrightarrow{\text{ev}} & I \\
\chi_{X,X^*} \big\downarrow & & \\
\overline{X^* \otimes X} & & \big\downarrow \mathring{\chi} \\
\overline{1_{X^*} \otimes i_X} \big\downarrow & & \\
\overline{X^* \otimes X^{**}} & \xrightarrow{\overline{\text{ev}_{X^*}}} & \overline{I}
\end{array}$$

The corresponding coevaluation is easily defined analogously.

Remark 12.54. The natural isomorphisms j, k and l are coherences since they are defined implicitly in terms of other coherences such as the monoidal or involutive ones, or the pivotal structure. Therefore, they will be invisible in the graphical calculus.

13 Half-twists and half-ribbon categories

One motivation for studying half-twists is to have a graphical calculus for ribbons performing a turn by π, as shown in Figure 13.1. This is possible in involutive monoidal categories if we represent the involution functor graphically by rotating the diagram by π: After a ribbon is rotated, its back is in the foreground, and a half-twist thus constitutes a morphism from the unrotated ribbon to the rotated one, or vice versa. (The chirality of the half-twist is a matter of convention, we follow [Egg11].) This graphical idea motivates large parts of this section.

The first subsection recalls the central results from the literature and supplements them by the new, but straightforward definition of "half-twist preserving functors".

In the second subsection, a key issue will be clarified that hasn't been addressed sufficiently by the founders of the field. It is the interplay between duals and half-twists. A rigid, balanced category can satisfy a nontrivial axiom, the "ribbon equation" (2.1.23). Algebraically, it fixes the value of the twist on dual objects. From the perspective of diagrams, it describes a compatibility between a turn of a ribbon by 2π (the twist) and a rotation of a diagram in the plane by π (dualisation). Since a half-twist is represented by a turn by π, we have to ask the question of a suitable compatibility law again.

A general theorem that verifies soundness of the resulting graphical calculus with respect to arbitrary isotopies of ribbons will be deferred to future work. We will be content with a suitable and natural axiom to impose on half-twists in the presence of duals, and use the graphical calculus merely as a notational tool to instantiate morphisms and lemmas. The new notion will then be "half-ribbon categories" (Definition 13.14), which have a half-twist and satisfy the ribbon axiom.

13.1 Half-twists and their graphical calculus

Definition 13.1 ([Egg11, Definition 4.3])**.** Let $\mathcal{C}$ be an involutive monoidal category, with involution $\overline{(\ )}$ and coherence morphisms χ and ε. A **half-twist** on $\mathcal{C}$ is a natural isomorphism $\tau\colon \overline{(\ )} \Rightarrow 1_{\mathcal{C}}$ such that the following two squares commute:

$$
\begin{array}{ccccc}
\overline{\overline{Y}\otimes\overline{X}}\otimes\overline{\overline{Z}} & \xleftarrow{\chi_{\overline{X},\overline{Y}}\otimes 1_{\overline{\overline{Z}}}} & \overline{\overline{X}}\otimes\overline{\overline{Y}}\otimes\overline{\overline{Z}} & \xrightarrow{1_{\overline{\overline{X}}}\otimes\chi_{\overline{Y},\overline{Z}}} & \overline{\overline{X}}\otimes\overline{\overline{Z}\otimes\overline{Y}} \\
{\scriptstyle \tau_{\overline{Y}\otimes\overline{X}}\otimes 1_{\overline{\overline{Z}}}}\downarrow & & \downarrow{\scriptstyle \tau_{\overline{X}}\otimes\tau_{\overline{Y}}\otimes\tau_{\overline{Z}}} & & \downarrow{\scriptstyle 1_{\overline{\overline{X}}}\otimes\tau_{\overline{Z}\otimes\overline{Y}}} \\
\overline{Y}\otimes\overline{X}\otimes\overline{\overline{Z}} & & \overline{X}\otimes\overline{Y}\otimes\overline{Z} & & \overline{\overline{X}}\otimes\overline{Z}\otimes\overline{Y} \\
{\scriptstyle 1_{\overline{Y}}\otimes\chi_{X,\overline{Z}}}\downarrow & & \downarrow{\scriptstyle \chi_{X,Y}\otimes 1_{\overline{Z}}} & & \downarrow{\scriptstyle \chi_{\overline{X},Z}\otimes 1_{\overline{Y}}} \\
\overline{Y}\otimes\overline{\overline{Z}\otimes X} & (T^l) & \overline{Y\otimes X}\otimes\overline{Z} & (T^r) & \overline{Z\otimes\overline{X}}\otimes\overline{Y} \\
{\scriptstyle 1_{\overline{Y}}\otimes\tau_{\overline{Z}\otimes X}}\downarrow & & \downarrow{\scriptstyle \chi_{Y\otimes X,Z}} & & \downarrow{\scriptstyle \tau_{Z\otimes\overline{X}}\otimes 1_{\overline{Y}}} \\
\overline{Y}\otimes\overline{Z}\otimes X & & \overline{Z\otimes Y\otimes X} & & Z\otimes\overline{X}\otimes\overline{Y} \\
{\scriptstyle \chi_{Y,Z}\otimes 1_X}\downarrow & & \downarrow{\scriptstyle \tau_{Z\otimes Y\otimes X}} & & \downarrow{\scriptstyle 1_Z\otimes\chi_{X,Y}} \\
\overline{Z\otimes Y}\otimes X & \xrightarrow[\tau_{Z\otimes Y}\otimes 1_X]{} & Z\otimes Y\otimes X & \xleftarrow[1_Z\otimes\tau_{Y\otimes X}]{} & Z\otimes\overline{Y\otimes X}
\end{array}
$$

The graphical representation of the axiom is given in Figure 13.2.

Remarks 13.2. In contrast to the original source [Egg11], the monoidal coherences have been strictified.

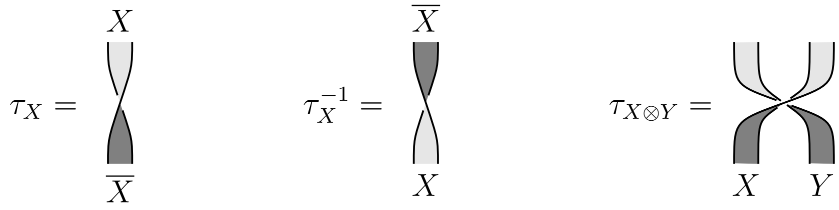


Figure 13.1: Graphical calculus of half-twists.

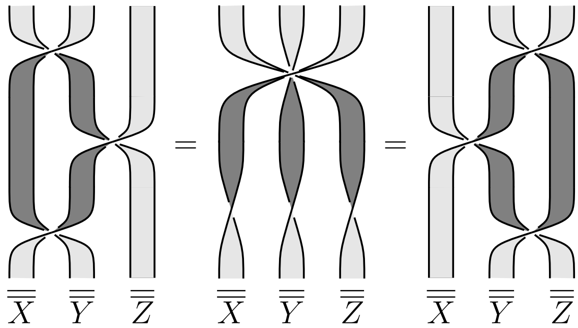


Figure 13.2: Egger's axioms T^l and T^r in diagrammatic calculus. To visualise the isotopy corresponding to the right hand equation, start with the diagram in the center, and pick up the right hand strand (labelled with Z) – which is behind the other strands – at its upper half-twist. Slide the upper half-twist down along the left strand (labelled with X) until it meets the lower half-twist of X. Then slide the lower half-twist of Z behind the lower half-twist of Y. For the left hand equation, an analogous isotopy exists.

In [Egg11], such a structure is called a "twist". We prefer the term "half-twist" from [ST09] and reserve "twist" for the structural isomorphism $\theta_X \colon X \to X$ from a balanced structure.

Lemma 13.3 ([Egg11, Lemma 4.2])**.** Let τ be a half-twist in an involutive monoidal category $\mathcal{C}$. Then for any object X in $\mathcal{C}$, the following holds:

$$\tau_{\overline{X}} = \overline{\tau_X} \qquad (\text{lem}_{\overline{\tau}})$$

Proof. Naturality of τ applied to τ_X itself implies:

$$\begin{array}{ccc} \overline{\overline{X}} & \xrightarrow{\overline{\tau_X}} & \overline{X} \\ \downarrow{\scriptstyle \tau_{\overline{X}}} & & \downarrow{\scriptstyle \tau_X} \\ \overline{X} & \xrightarrow{\tau_X} & X \end{array}$$

Since τ_X is invertible, the lemma follows. □

Remark 13.4 ([Egg11, Remark 4.6]). If we assume that the correct graphical representation of τ is a half-twist, then the graphical representation of $\overline{(\ )}$ must be rotation by π. To rule out the graphical representation by reflection, simply consider the statement of the previous lemma, which would be falsified graphically.

To the knowledge of the author, half-twists have been used as graphical representations of algebraic data only in [ST09], [Sel10] and [Egg11].

Lemma 13.5 ([Egg11, Lemma 4.2]). In an involutive monoidal category $\mathcal{C}$ with a natural transformation $\tau \colon \overline{(\ )} \to 1_{\mathcal{C}}$, the two axioms T^r and T^l for τ are equivalent.

Thus in the definition of the half-twist, we could choose only one of the two.

Proof. A full proof is given in the cited article. As the key step, the involution functor is applied to one of the axioms to rotate it by π and arrive at the other axiom. □

Graphical representations of the half-twist on a single strand, its inverse, and a half-twist on two strands can be seen in Figure 13.1. For the half-twist of two strands, the diagrammatic calculus forces the left strand to cross the right strand, very akin to a braiding. Indeed, the only difference to a braiding are the individual twists on each strand. It therefore seems reasonable to expect that a braiding can be defined by a half-twist, and this is verified by the following theorem.

Theorem 13.1 ([Egg11, Theorem 4.4]). Given an involutive monoidal category with half-twist, a balanced structure $(c_{X,Y} \colon X \otimes Y \to Y \otimes X, \theta_X \colon X \to X)$ can be defined by the following commuting diagrams:

$$\begin{array}{ccc} \overline{X} \otimes \overline{Y} & \xrightarrow{\chi_{X,Y}} & \overline{Y \otimes X} \\ {\scriptstyle \tau_X \otimes \tau_Y}\downarrow & & \downarrow{\scriptstyle \tau_{Y \otimes X}} \\ X \otimes Y & \xrightarrow{c_{X,Y}} & Y \otimes X \end{array} \tag{def$_c$}$$

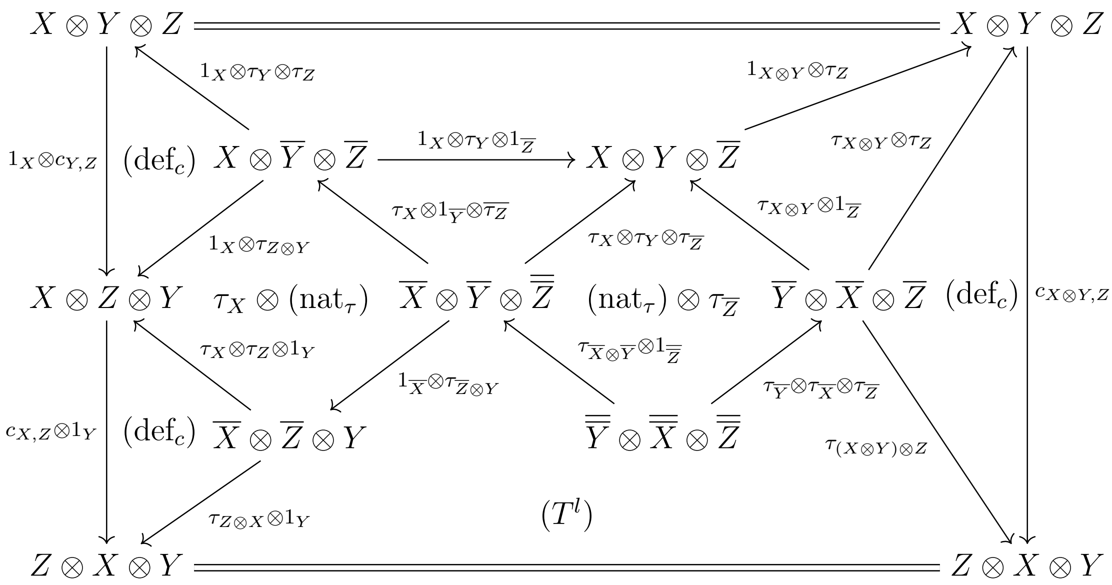
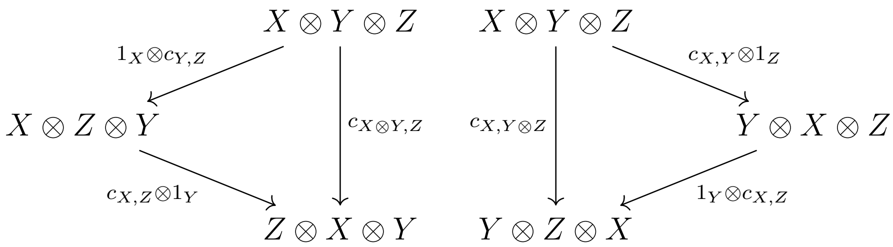


Figure 13.3: A braid law can be proven each from (T^l) and (T^r). The case for (T^l) is shown here in the strictified setting, i.e. $\overline{X \otimes Y} = \overline{Y} \otimes \overline{X}$ and χ is the identity.

$$\begin{array}{ccc} \overline{\overline{\overline{X}}} & \xrightarrow{\tau_{\overline{X}}} & \overline{X} \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \tau_X} \\ X & \xrightarrow{\theta_X} & X \end{array} \tag{def$_\theta$}$$

This balanced structure is compatible with the involutive structure in the sense of Definition 12.41.

Proof. Since a detailed proof is missing in the literature, we will work through it. Three axioms have to be proven: The balance equation ($\mathrm{ax}_{c\theta}$) (page 77), and the two braid axioms (2.1.18). The latter two are reproduced here, after monoidal strictification:



($\mathrm{ax}_{c\otimes}$)

The two braid axioms will follow from (T^r) and (T^l), respectively. The proof for one of them is given in Figure 13.3, the other one is analogous. The proof is probably

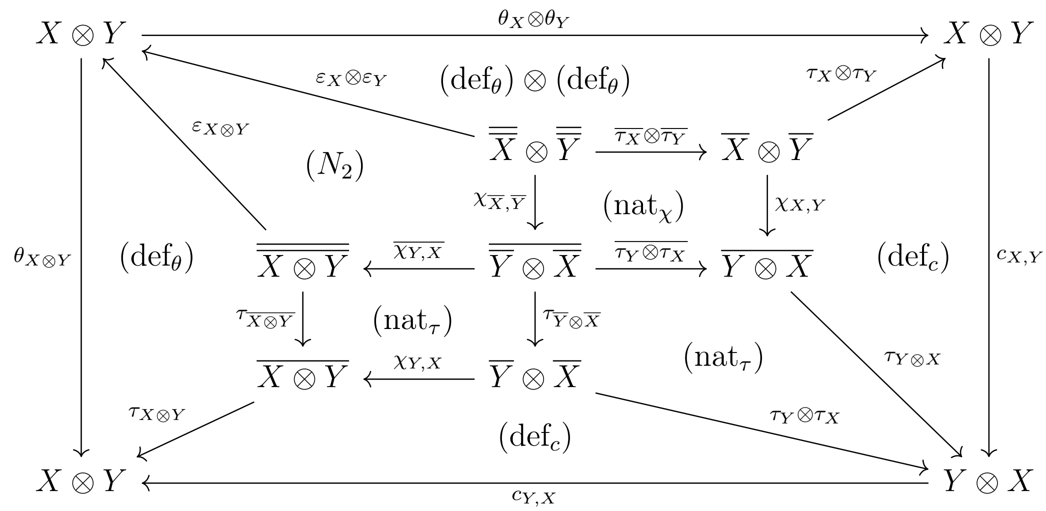


Figure 13.4: The balance law for twist and braiding from a half-twist follows naturally. It is interesting to note that the coherence isomorphisms cancel through (N_2) from page 82, which itself resembles the balance law.

not too surprising since the definition of the braiding (def_c) from page 98 as well as (T^r) and (T^l) from page 96 were designed to reproduce the braid axioms. The more interesting insight is maybe that (T^r) and (T^l) imply each other, so finding half-twists is in a sense a more efficient procedure than finding braidings, since fewer axioms have to be checked.

The key idea for the balance equation is that it is essentially the square of (def_c). The balance equation states that the square of the braiding is the twist of both objects divided by the twists of the individual objects. Correspondingly, the very definition of the braiding states that it is the half-twist of both objects divided by the individual half-twists. The proof is seen in Figure 13.4. The new braiding satisfies $(\mathrm{ax}_{\overline{c}})$ and is therefore compatible with the involutive structure, by definition of the braiding and naturality, as is seen in Figure 13.5. The proof of $(\mathrm{ax}_{\overline{\theta}})$ is left as an exercise. □

Definition 13.6 ([Egg11, Definition 4.3]). An **involutive half-twist** is a half-twist satisfying $\varepsilon \circ \overline{\tau} \circ \tau = 1_{1_{\mathcal{C}}}$.

In a category with involutive half-twist, each object is naturally equipped with an involution.

Lemma 13.7 ([Egg11, Corollary 4.5]). If a half-twist is involutive, then the balanced

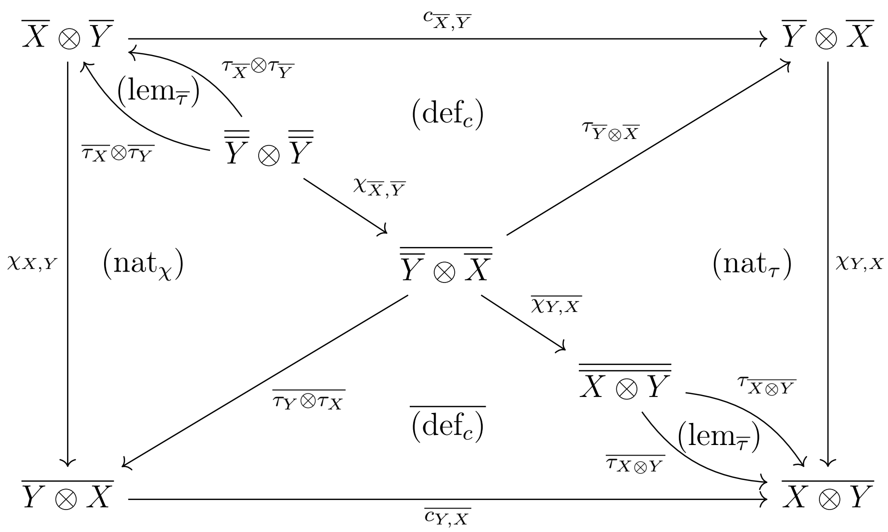


Figure 13.5: The braiding defined by a half-twist is always involutive.

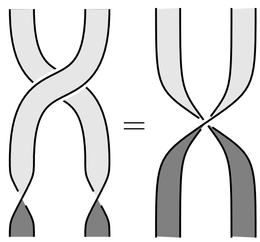


Figure 13.6: Definition (def_c) of the braiding in terms of the half-twist.

structure arising from it is in fact a symmetry.

Proof. By definition, the twist θ is the identity. □

Remark 13.8. The graphical representation of τ as a half-twist is perfectly compatible with the representations for braiding and twist, as is seen in Figure 13.6.

Remark 13.9 ([Egg11, Remark 4.6]). "Star bar categories" [BM09, Definition 4.5] are a superficially similar, but different concept than half-twists.

"Self-duality structures" [Sel10] seem closely related to half-twists, although working out the exact correspondence remains for future work.

Remark 13.10. In [Enr10], "half-balanced" categories are introduced, which – in the present language – amount to natural isomorphisms $a\colon 1_{\mathcal{C}} \implies \overline{(\ )}$ in a balanced monoidal category with strict involution, satisfying (def_c), but not necessarily (T^r) or (T^l).

It remains to be seen whether important counterexamples arise that are "half-balanced" without a^{-1} being a half-twist. Furthermore, any graphical intuition seems to be missing or obscured in the cited article.

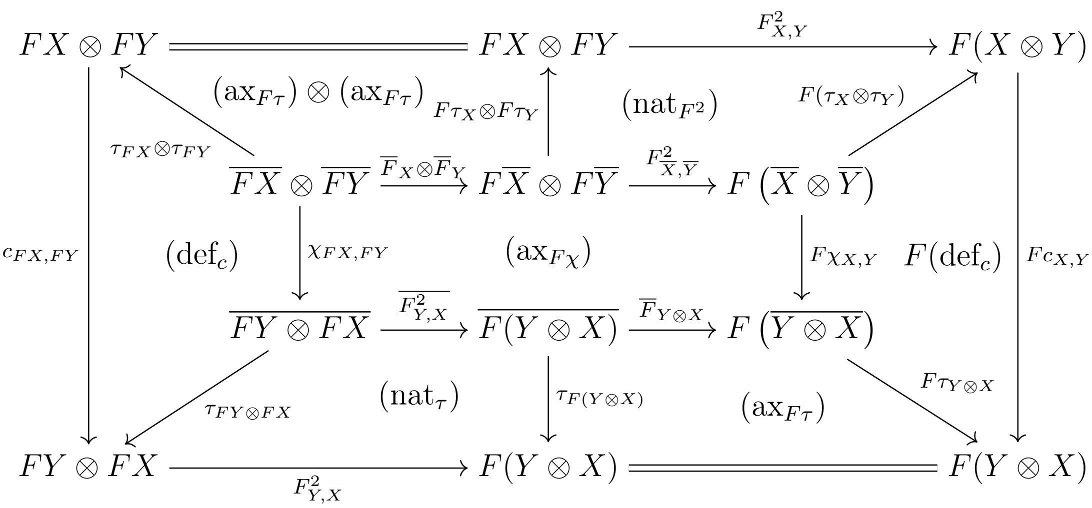
For the next definition and lemma, recall that a monoidal functor F has coherence morphisms $F^2_{X,Y}: FX \otimes FY \to F(X \otimes Y)$ and $F^0: I_{\mathcal{D}} \to FI_{\mathcal{C}}$, and an involutive monoidal functor has an additional coherence morphism $\overline{F}_X: \overline{FX} \to F\overline{X}$, as described in Definition 12.16.

Definition 13.11. Let $F: \mathcal{C} \to \mathcal{D}$ be an involutive monoidal functor, τ a half-twist on $\mathcal{C}$ and, by overloading of the same symbol, τ a half-twist on $\mathcal{D}$. F is said to be **half-twist preserving** if the following diagram commutes:

$$\begin{array}{ccc} \overline{FX} & \xrightarrow{\tau_{FX}} & FX \\ \downarrow{\scriptstyle \overline{F}_X} & & \| \\ F\overline{X} & \xrightarrow{F\tau_X} & FX \end{array} \qquad (\mathrm{ax}_{F\tau})$$

Lemma 13.12. Let $F: \mathcal{C} \to \mathcal{D}$ be involutive monoidal and half-twist preserving. Then F is a balanced (i.e. it preserves braiding and twist) functor for the balanced structures coming from the half-twists.

Proof. The proof for preservation of the braiding is presented in the following commutative diagram, while the proof for preservation of the twist is left as an exercise.



□

Axiom ($\mathrm{ax}_{F\chi}$) from page 85 plays a key role.

13.2 Half-ribbon categories

A ribbon category is a category with duals and a balanced structure, and an axiom which the twist must satisfy. The pressing question is therefore: In the presence of duals, what axioms must a half-twist satisfy in order for the resulting twist to be ribbon? A category satisfying the correct axiom would be called a *half-ribbon category.*

We have seen now that a half-twist gives rise to a balanced structure. It is well-known (Theorem 2.1) that in a rigid, braided category, twists and pivotal structures are in (noncanonical) correspondence. It will turn out to be more convenient to consider the pivotal structure defined by the balanced structure to understand half-ribbon categories.

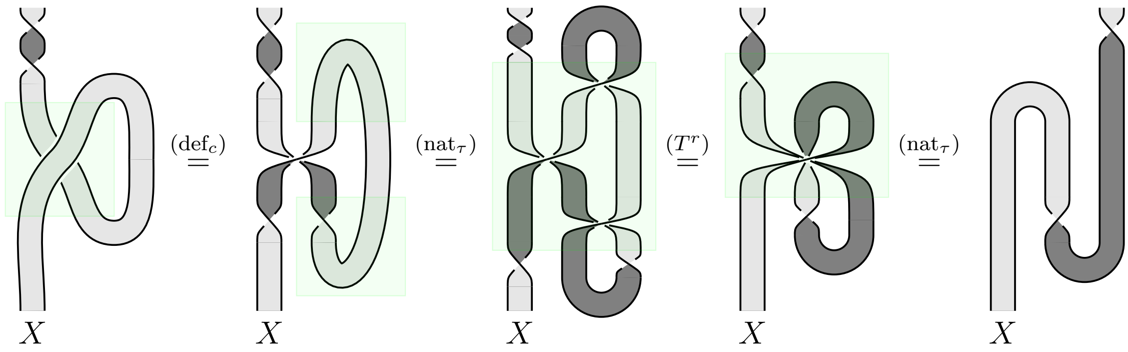
Lemma 13.13. Assume an involutive monoidal category with half-twist that is also rigid, i.e. has right and left duals. The pivotal structure from Theorem 2.1, with respect to the balanced structure from the half-twist, has the following form:

$$\begin{aligned} X &\xrightarrow{(1_X\otimes\mathring{\chi})\circ\rho_X^{-1}} X\otimes\overline{I} \xrightarrow{1_X\otimes\mathrm{coev}_{X^*}} X\otimes\overline{X^{**}\otimes X^*} \\ &\xrightarrow{1_X\otimes\chi^{-1}_{X^*,X^{**}}} X\otimes\overline{X^*}\otimes\overline{X^{**}} \xrightarrow{1_X\otimes\tau_{X^*}\otimes 1_{\overline{X^{**}}}} X\otimes X^*\otimes\overline{X^{**}} \\ &\xrightarrow{\mathrm{ev}_X\otimes 1_{\overline{X^{**}}}} I\otimes\overline{X^{**}} \xrightarrow{\lambda_{\overline{X^{**}}}} \overline{X^{**}} \xrightarrow{\varepsilon_{X^{**}}\circ\tau^{-1}_{\overline{X^{**}}}} X^{**} \end{aligned} \tag{13.2.1}$$

Note that it only contains half-twists on single strands.

Proof. See Figure 13.7, which contains a diagrammatic proof. The left hand side is derived from solving (2.1.22) for the pivotal structure. We first insert the definition of the braiding in terms of a half-twist of two strands. We apply naturality of τ to the evaluations and coevaluations. The key step now consists of using the axiom (T^r) to join three half-twists of two strands to a single half-twist of three strands and three individual half-twists (of which the outer ones cancel). The half-twist of three strands is then cancelled with the lower of the top two half-twists, yielding the result on the right hand side, which equals (13.2.1) after close inspection. □

Note that we have not used arbitrary three-dimensional isotopies, but only those moves justified by their algebraic counterparts from the earlier sections.



(The green rectangles mark the area where the diagram will be manipulated next.)

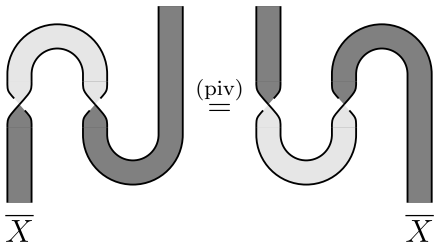
Figure 13.7: A pivotal structure arising from a half-twist. (Other similar constructions exist and give rise to potentially different pivotal structures.)

Definition 13.14. A **half-ribbon category** is a rigid involutive monoidal category with half-twist that is involutive pivotal, with respect to the pivotal structure from Lemma 13.13.

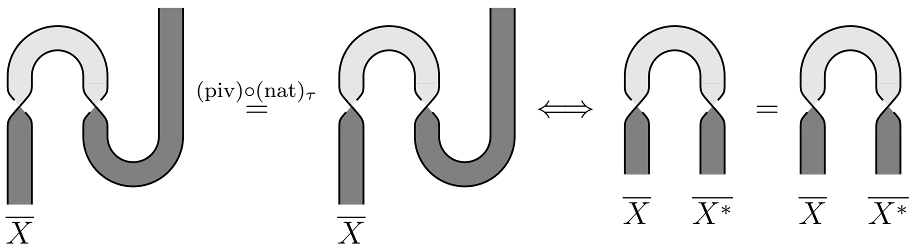
The graphical representation for the pivotality condition is found in Figure 13.8a.

Remark 13.15. It is worth expanding this definition. Given a pivotal category $\mathcal{C}$ with pivotal structure i, its reverse $\mathcal{C}^{\text{rev}}$ has a canonical pivotal structure which is the inverse of i (Definition 12.46). If $\mathcal{C}$ is involutive pivotal, the involution functor $\overline{(\ )}$ is required to preserve the pivotal structure (Definition 12.50). In a rigid category, a half-twist noncanonically determines a pivotal structure, we pick the one from Lemma 13.13 for graphical reasons. If we choose the same half-twist on $\mathcal{C}^{\text{rev}}$, the involution functor preserves it (Lemma 13.3). The half-twist on $\mathcal{C}^{\text{rev}}$ defines a pivotal structure again by Lemma 13.13, onto which i is mapped by the involution functor. However, the choice is noncanonical, and usually it is not equal to the pivotal structure arising from the reversal of the monoidal product. In a half-ribbon category, however, all of the different choices are required to coincide.

Lemma 13.16. A rigid involutive monoidal category with half-twist is a half-ribbon



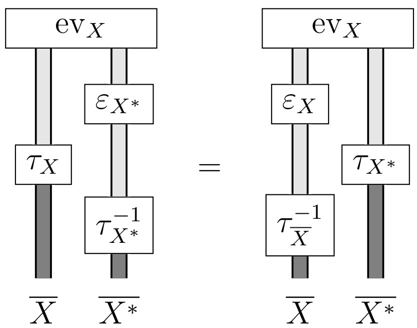
(a) The involution functor is required to be pivotal, (piv) from page 94. On the left hand side, we see the inverse $\iota_{\overline{X}}$ of the pivotal structure. It is equated to $\overline{i_X}$, which is the result from Figure 13.7 rotated by π.



(b) Applying naturality of the half-twist to the right hand side of Figure 13.8a yields the first equation. Due to the snake identities (2.1.3), the coevaluation can be removed, yielding the second, more concise equation on the right. For both equations, the two sides resemble each other closely, but note the subtle difference in the turn directions of the half-twist.

Figure 13.8: In a half-ribbon category, the involutive structure is required to be pivotal.

category iff the half-twist satisfies the following equation for every object X:



(13.2.2)

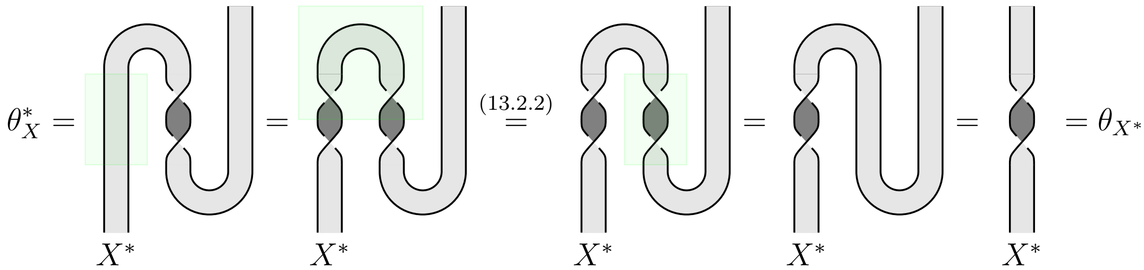
Proof. The proof is shown graphically in Figure 13.8b. First, naturality of the left half-twist in the right hand side of Figure 13.8a is applied to yield the diagrammatic equation on the left.

Now, the coevaluations can be removed in both sides of the resulting equation, because of the snake identities (2.1.3). This yields the claim. □

A ribbon category is a balanced, rigid category with an axiom relating the balance to the duals. Since twists in a braided, rigid category are in bijective correspondence with pivotal structures (Theorem 2.1), we can as well read the ribbon axiom as an axiom on the pivotal structure. The half-ribbon axiom has of course been chosen such that it implies the ribbon axiom:

Proposition 13.17. In a half-ribbon category, the twist satisfies the ribbon equation (2.1.23).

Proof. We will again proceed graphically:



First, we introduce a cancelling half-twist pair on the definition of the dual twist. We can now use Lemma 13.16, which follows from pivotality. The remaining steps consist of cancelling a further half-twist pair and invoking the snake identity to arrive at the twist of the dual. □

13.3 Half-twists in †-categories

When moving from ordinary categories to †-categories, one usually requires all coherence morphisms, as well all structural morphisms, to be unitary, although for separate reasons.

Coherence morphisms only deserve their name if any two compositions of coherences between two fixed objects coincide. Coherences are then required to be unitary in order to prevent the †-structure from introducing new morphisms.

Structural morphisms like braidings and twists are often required to be unitary in order to validate laws which are true in the graphical calculus (or which are desirable for other reasons). Half-twists will be no exception here.

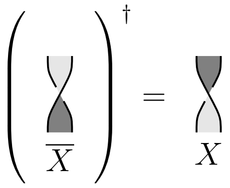


Figure 13.9: Demanding the half-twist to be unitary validates an obvious graphical law: The horizontal reflection of a half-twist is its inverse.

Definition 13.18. An **involutive monoidal †-category** is an involutive monoidal category that is also a †-category, such that the coherence isomorphisms ε and χ are unitary.

Definition 13.19. A half-twist in an involutive monoidal †-category is always required to be unitary. The graphical representation of this law is Figure 13.9.

Remark 13.20. The graphical calculus of †-categories is compatible with the graphical calculus of half-twists, i.e. the reflection of a half-twist at the horizontal plane is its inverse.

Unsurprisingly, the braiding and twist defined by a unitary half-twist are again unitary.

13.4 Examples

We give an informal account of the category $\mathcal{HRIB}$ defined in [ST09], which realises the graphical calculus of half-twists.

Definition 13.21 ([ST09, Definition 4.5])**.** For a set S, the half-ribbon category $\mathcal{HRIB}(S)$ is defined as follows.

Objects Sets of finitely many closed intervals on the real line, each equipped with a direction (up or down), an element of S as label, and an orientation of the thickening of the interval. We encode the orientation as a shading: Light for the orientation agreeing with a standard orientation of the thickening of the real line, and dark for the opposite orientation.

Morphisms Isotopy equivalence classes of tangles of oriented, directed, S-labelled ribbons embedded in $\mathbb{R}^2 \times [0, 1]$. The first component is the horizontal axis in the plane of drawing, the second component is the horizontal axis perpendicular to it. The third component is thought of as a height, and will also be the direction of composition. The ribbons may be closed annuli, or diffeomorphic to $[0,1] \times [0,1]$. In the latter case, the ends of the ribbons are required to end on $\mathbb{R} \times \{0\} \times \{0,1\}$. Source and target are given by projection of the ends onto 0 and 1 in the third component of the embedding, respectively. Composition is given by identifying the tangles along the common real line and rescaling. The identity morphism to a given object is its product with the interval.

Monoidal structure The monoidal product is given by horizontal juxtaposition of intervals and tangles. The monoidal unit is the empty set.

Involutive structure The involution functor rotates a tangle and the intervals by π around the vertical axis (and therefore interchanges light and dark shadings). The coherence isomorphisms χ and ϵ are then mere witnesses of the geometrical facts that a rotation by π reverses the order of objects and a rotation 2π is equal no rotation.

Half-twist One or several ribbons turning by π.

†-structure Reflection on the horizontal plane, or equivalently postcomposition of each ribbon parametrisation with the map $(x, y, z) \mapsto (x, y, 1 - z)$.

Remark 13.22. This definition is slightly amended with respect to [ST09, Definition 4.5]. Here, we don't deem it necessary to consider objects up to isotopy. (Instead, isotopic objects are isomorphic via ribbons implementing the isotopy.) We also detail the involutive monoidal structure and the half-twist – a concept which was not available then to Snyder and Tingley – and the †-structure.

Example 13.23. Let $\mathrm{Hilb}_{\mathbb{R},f.d.}$ be the symmetric monoidal category of finite dimensional real inner product vector spaces. It has a natural involutive monoidal structure with half-twist:

$\mathrm{Hilb}_{\mathbb{R},f.d.}$ is a pivotal †-category, and the involution structure is given by the covariant dual functor, as in Lemma 12.36. The †-structure for $f\colon V \to W$ arises from the inner product via $\left\langle f^\dagger v, w\right\rangle_W = \langle v, fw\rangle_V \,\forall v \in V, w \in W$. A dualising counit for the inner product can always be found since the vector spaces are finite dimensional.

For a given vector space V with inner product $\langle -, -\rangle_V : V \otimes V \to \mathbb{R}$, the half-twist $\tau_V\colon V^* \to V$ is given by $\tau_V^{-1}(v) := \langle v, -\rangle_V$.

With suitable definitions of tensor product and dual inner product space, this half-twist is involutive (Definition 13.6), and $\mathrm{Hilb}_{\mathbb{R},f.d.}$ becomes a half-ribbon category giving rise to the original symmetric monoidal structure.

Counterexample 13.24. Consider the involutive monoidal category $\mathrm{Vect}_{\mathbb{C}}$ (Example 12.11), where involution is complex conjugation. There exists *no* half-twist for this balanced involutive structure.

Proof. Assume τ is a half-twist for $\mathrm{Vect}_{\mathbb{C}}$ giving rise to the standard symmetric braiding and the trivial twist. Note that then we have $\tau \circ \overline{\tau} = \varepsilon$, and $\mathring{\chi}$ satisfies $\mathring{\chi}^{-1} \circ \overline{\mathring{\chi}^{-1}} = \varepsilon_I$ [Egg11, (N_0) from Lemma 2.3]. We can therefore infer $\tau_I = \pm \mathring{\chi}^{-1}$. The positive sign will be assume for now, but the proof for the negative sign choice is analogous.

Let now V be a complex vector space and $v \in V$. Then $\lambda \mapsto \lambda v$ is a linear map $\mathbb{C} \to V$. Its naturality square for τ is then:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathring{\chi}} & \overline{\mathbb{C}} \\ {\scriptstyle \lambda \mapsto \lambda v}\downarrow & & \downarrow{\scriptstyle \lambda \mapsto \lambda v} \\ V & \xrightarrow{\tau_V^{-1}} & \overline{V} \end{array}$$

It implies that the maps $\lambda \mapsto \tau_V^{-1}(\lambda v) = \lambda \tau_V^{-1}(v)$ and $\lambda \mapsto \mathring{\chi}(\lambda) v = \overline{\lambda} v$ coincide. In particular, they must coincide for $\lambda = 1$, which implies $\tau_V^{-1}(v) = v$ for each $v \in V$. But this is not a linear map since $\overline{V}$ is equipped with the conjugate scalar multiplication, a contradiction. □

This counterexample was found by Egger (private conversation) and the author independently.

13.5 Strictification of the half-twist

I thank the participants of the mathematical physics seminar in Vienna, in particular Gregor Schaumann, for an inspiring discussion after which I could write this subsection.

Lemma 13.25. Let $\mathcal{C}$ be a balanced category. Choose the braiding c and the twist θ as the involutive coherences, as in Example 12.15. Then $1_{1_\mathcal{C}}$ is a half-twist for it. It gives rise to the original balanced structure c and θ.

Proof. Since $\tau_X = 1_X$, the axiom (T^r) simplifies considerably. The remaining condition on $\chi = c$ is easily proved via naturality of c and a braid axiom. It is easy to check that in this case, the same braiding and twist arise as before. □

Definition 13.26. Let $(\mathcal{C}, - \otimes -, I, \overline{(\ )}, \chi, \epsilon)$ be an involutive monoidal category with half-twist τ. Denote the resulting braiding and twist by c and θ, respectively.

The involutive monoidal category $\mathcal{C}_t$ is defined as $(\mathcal{C}, - \otimes -, I, 1_\mathcal{C}, c, \theta)$, equipped with the trivial half-twist $1_{1_\mathcal{C}}$. Note that the involutive structure consists of c and θ, as in Example 12.15 and the previous lemma.

The half-twist is then said to be **strictified**.

Proposition 13.27. Let $\mathcal{C}$ be an involutive monoidal category with half-twist. There is a half-twist preserving involutive monoidal equivalence between $\mathcal{C}$ and $\mathcal{C}_t$.

Proof. We adopt the notation from the definition of $\mathcal{C}_t$ and directly construct two inverse involutive monoidal functors $F : \mathcal{C} \rightleftarrows \mathcal{C}_t : G$. The underlying monoidal functors are each the identity functor. As involutive coherence morphism $\overline{F}$, we choose $\overline{F}_X = \tau_X$. Recalling that the mere functor F and the monoidal coherence F^2 are trivial, we reduce ($\mathrm{ax}_{F\chi}$) and ($\mathrm{ax}_{F\varepsilon}$) from page 85 to the definition of c and θ, respectively. The coherence $\overline{G}$ is τ^{-1}, which makes G the (strict) inverse of F. □

Remark 13.28. After discovering that half-twists lead to balanced structures, it is a natural question to ask which balanced structures come from half-twists. The previous proposition shows that, in a sense, all do. But the argument required the choice of an unnatural involutive structure. Furthermore, we saw in Example 13.24 that certain involutive monoidal structures do not admit half-twists at all, which suggests that the involutive structure is part of the data on which the existence question of a half-twist should depend.

We should thus ask: "Given a balanced involutive monoidal category, is there a half-twist for the involutive structure that reproduces the given balanced structure?" In all generality, this question is open. Possibly, an obstruction theory can be developed.

13.6 Half-twisted categories

This subsection tries to give a maximal answer to the question whether a given balanced involutive monoidal categories admits a half-twist. The key idea is simply to select all objects that are isomorphic to their conjugate, together with a choice of isomorphism that squares to the twist. These objects are then shown to form a category. Its remaining structure is constructed with the aim to model the original balanced involutive structure as closely as possible.

Definition 13.29. Let $\mathcal{C}$ be a balanced involutive monoidal category. The **half-twisted category** $\mathcal{C}_{\frac{1}{2}}$ consists of:

Objects Tuples $\left(X, \zeta\colon \overline{X} \xrightarrow{\cong} X\right)$ satisfying:

$$\zeta \circ \overline{\zeta} = \theta_X \circ \varepsilon_X \tag{ax$_{\zeta^2}$}$$

Morphisms A morphism $f\colon (X_1, \zeta_1) \to (X_2, \zeta_2)$ is a morphism $f\colon X_1 \to X_2$ commuting with ζ_1 and ζ_2 in the following way:

$$\begin{array}{ccc} \overline{X_1} & \xrightarrow{\overline{f}} & \overline{X_2} \\ {\scriptstyle\zeta_1}\downarrow & & \downarrow{\scriptstyle\zeta_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array} \tag{ax$_{f\zeta}$}$$

Monoidal unit It is defined as $(I, \mathring{\chi}^{-1})$, with $\mathring{\chi}$ from (def$_{\mathring{\chi}}$).

Monoidal product Define $(X_1, \zeta_1) \otimes (X_2, \zeta_2) := (X_1 \otimes X_2, \zeta_{12})$, such that:

$$\begin{array}{ccc} \overline{X_2} \otimes \overline{X_1} & \xrightarrow{\zeta_2 \otimes \zeta_1} & X_2 \otimes X_1 \\ {\scriptstyle\chi_{X_2,X_1}}\downarrow & & \downarrow{\scriptstyle c_{X_2,X_1}} \\ \overline{X_1 \otimes X_2} & \xrightarrow{\zeta_{12}} & X_1 \otimes X_2 \end{array} \tag{def$_{\zeta_{12}}$}$$

Involution functor On objects, we define straightforwardly:

$$\overline{(X, \zeta)} := \left(\overline{X}, \overline{\zeta}\right) \tag{def$_{\overline{\zeta}}$}$$

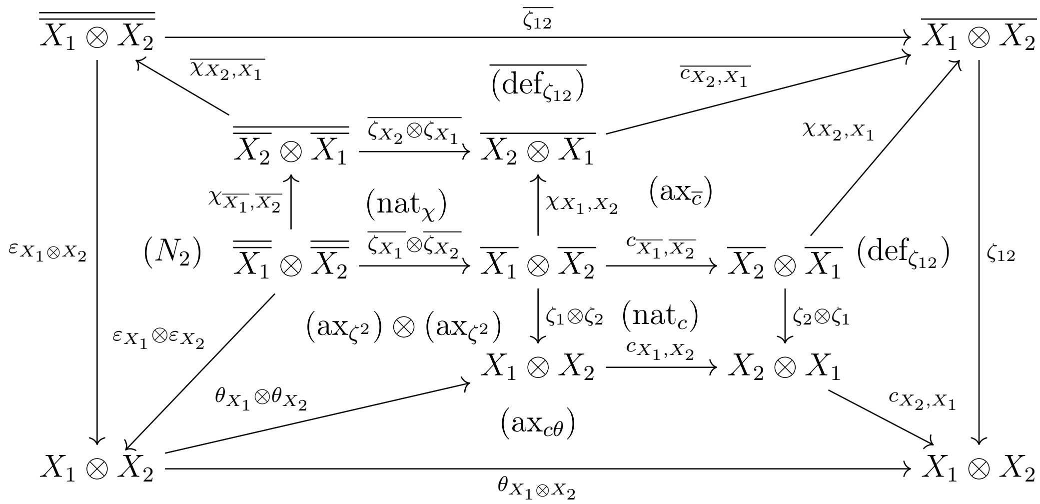
On morphisms, the involution functor is as before.

Involution coherences $\chi_{(X_1,\zeta_1),(X_2,\zeta_2)} := \chi_{X_1,X_2}$ and $\varepsilon_{(X,\zeta)} := \varepsilon_X$ are again coherence morphisms for the new involutive structure.

Half-twist We can finally define the half-twist as $\tau_{(X,\zeta)} \coloneqq \zeta$.

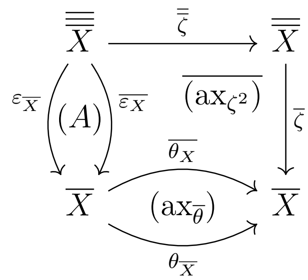
Lemma 13.30. The previous definition constitutes an involutive monoidal category with half-twist.

Proof. **Monoidal product** ζ_{12} satisfies (ax_{ζ^2}):



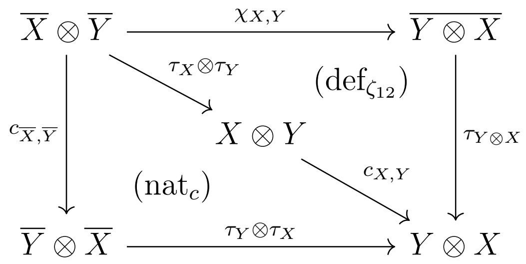
We relied on $(\mathrm{ax}_{c\theta})$ from page 77.

Involution functor The definition of the involution on objects satisfies (ax_{ζ^2}):



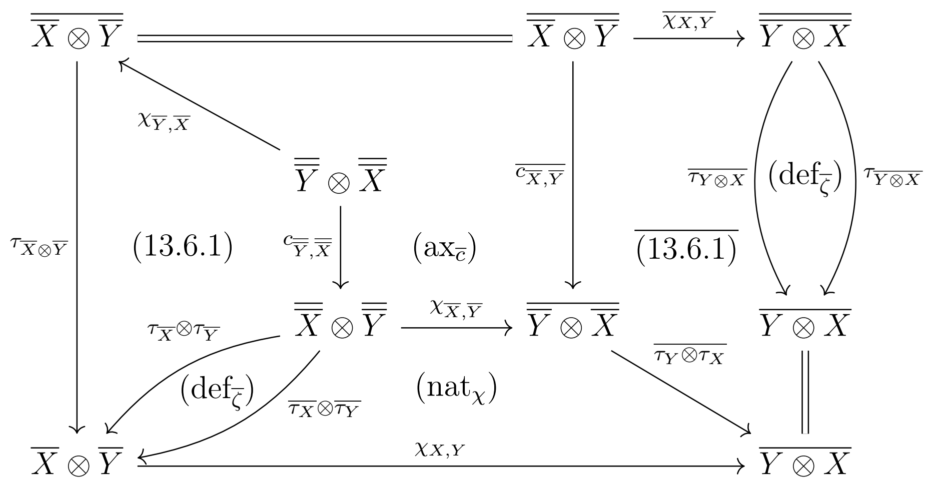
The involution on morphisms satisfies $(\mathrm{ax}_{f\zeta})$ by applying $\overline{(\ )}$ to $(\mathrm{ax}_{f\zeta})$ itself and inserting the definition of $\overline{(\ )}$ on objects.

Involution coherences χ and ε are morphisms in $\mathcal{C}_{\frac{1}{2}}$. First, note this lemma:

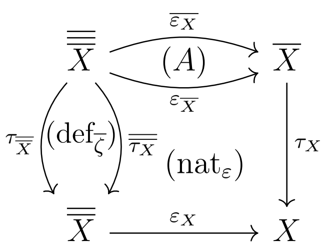


(13.6.1)

Using the lemma, it can be shown that χ satisfies (ax$_{f\zeta}$):

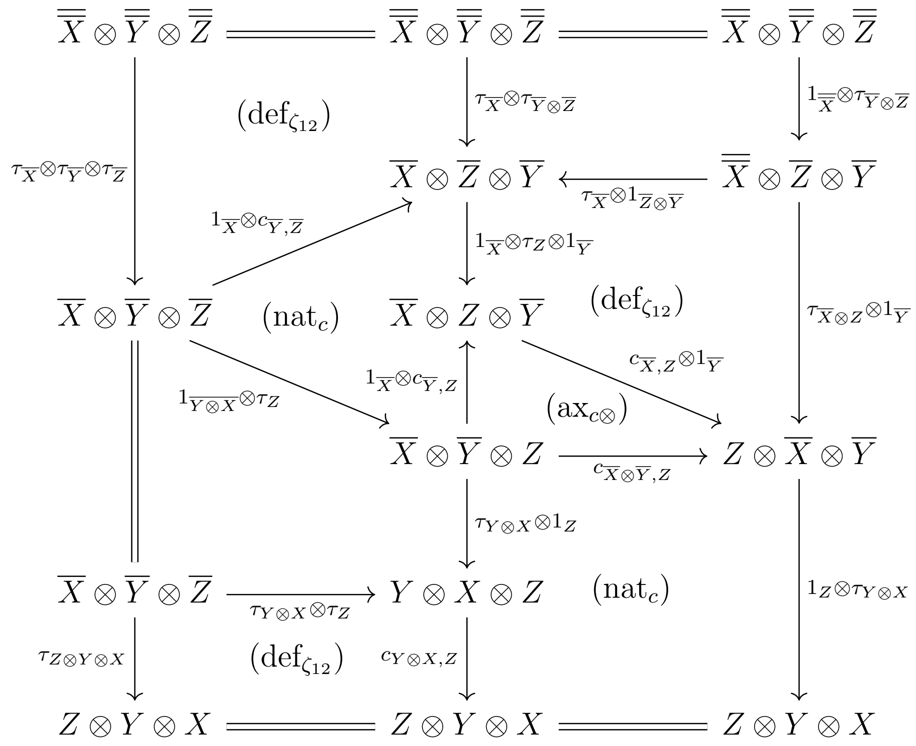


ε does as well:



Half-twist τ defined previously is a valid half-twist, that is, it satisfies (T^r). In order to condense the proof to a readable size, it has been strictified, i.e. $\overline{X \otimes Y} = \overline{Y} \otimes \overline{X}$ and χ is the identity. (ε does not appear in the proof.) The heartpiece is the compatibility of the braiding with the monoidal product,

($\mathrm{ax}_{c\otimes}$) (which can be found on page 99).



□

Remark 13.31. The philosophy behind this construction is very akin to *equivariantisation* (see e.g. [BN13, Section 2.1] for an introduction) but there are some subtle differences.

In the equivariantisation of the $\mathbb{Z}_2$-action given by $\overline{(\ )}$ and ε (Remark 12.8), we would study tuples $(X, \zeta\colon \overline{X} \to X)$ satisfying $\overline{\zeta}\zeta = \varepsilon_X$, corresponding to the special case of the half-twisted category with trivial twist (and thus symmetric braiding). Furthermore, group actions on monoidal categories are usually required to be monoidal automorphisms, whereas $\overline{(\ )}$ is an anti-automorphism.

If one is willing to restrict to symmetric categories where the braiding is a coherence, the differences between equivariantisation and the half-twisted category seem negligible, and one can adopt a definition like the "category of self-conjugates" in [Jac12, Definition 3.1].

Definition 13.32. There is a canonical forgetful functor $U\colon \mathcal{C}_{\frac{1}{2}} \to \mathcal{C}$, which is involutive monoidal. Its map on objects is $U(X,\zeta) = X$, and its map on morphisms is the identity. The monoidal and involutive coherences are all identities.

Lemma 13.33. U is balanced, i.e. the braiding and twist induced by τ maps to the braiding and twist of $\mathcal{C}$ under the forgetful functor U.

Proof. The axiom on the objects (ax_{ζ^2}) mirrors the definition of the twist (def_θ) from a half-twist, see page 99. Therefore, the twist of $\mathcal{C}_{\frac{1}{2}}$ is $\tau \circ \overline{\tau} \circ \varepsilon^{-1} = \theta$. Similarly, the monoidal product ($\text{def}_{\zeta_{12}}$) has been defined exactly like (def_c), such that it leads to the braiding c. ☐

Definition 13.34. Let $\mathcal{C}$ be a ribbon category with chosen duals that is also pivotal involutive. We define again the half-twisted category $\mathcal{C}_{\frac{1}{2}}$ as in Definition 13.29. We pull back the pivotal structure along the forgetful functor, i.e. $i_{(X,\zeta)} \coloneqq i_X$, and choose the following duals:

$$(X,\zeta)^* = \left(X^*, (\varepsilon_X^*)^{-1}\overline{\zeta}^* j_X\right) \qquad (\text{def}_{\zeta^*})$$

j_X is taken from Definition 12.52.

Lemma 13.35. The half-twisted category $\mathcal{C}_{\frac{1}{2}}$ is a half-ribbon category.

Proof. First, one has to show that $(X,\zeta)^*$ as defined above is indeed a dual for (X,ζ), i.e. the evaluation and coevaluation morphisms ev_X and coev_X satisfy ($\text{ax}_{f\zeta}$) and thus are indeed morphisms in $\mathcal{C}_{\frac{1}{2}}$. This follows from inserting Definitions ($\text{def}_{\zeta_{12}}$) and (def_j) (page 95), and then recognising the definition of the twist in terms of the pivotal structure and the braiding, (2.1.22).

Second, it has to be shown that $\mathcal{C}_{\frac{1}{2}}$ is pivotal involutive for the pivotal structure coming from the half-twist. By assumption, $\mathcal{C}$ already is. But since the forgetful functor U is balanced, and the pivotal structure i is pulled back along it, the axiom (piv) is satisfied in $\mathcal{C}_{\frac{1}{2}}$ as well. ☐

14 Outlook

A number of open questions about half-twists remain to be studied.

It has been shown in Section 13.5 that any balanced structure comes from a half-twist if we can choose the involutive structure freely. However, if the involutive structure is fixed as well, there are counterexamples (13.24). It should be studied which balanced involutive categories have half-twists.

From any monoidal category, we can construct a canonical braided monoidal category, the Drinfeld centre [Maj00, Corollary 9.1.6]. The question whether from any involutive monoidal category we can define a canonical category with half-twist suggests itself. The construction of half-twisted categories from Section 13.6 is a step in that direction, but it requires a balanced involutive monoidal category.

Braided monoidal categories have algebraic counterparts in the form of quasitriangular Hopf algebras. For deformed universal enveloping algebras of complex simple Lie algebras, the algebraic formulation of half-twists exists [ST09] and has in parts founded the field. A generalisation applicable to arbitrary Hopf algebras is under preparation by the author and will be published separately.

The precise relation between Egger's definition of half-twists and Selinger's self-duality structures [Sel10] remains to be established.

While plenty of examples for half-ribbon categories have been presented, none of them are entirely new since the resulting braidings were known already. An astonishing result would be a category with a new half-twist that leads to a previously unknown braided monoidal category. But since many examples for braidings are already known, the chances for such a result seem low.

How can we use half-twists and their graphical calculus? A first thought that might come to mind would be to evaluate a nonoriented ribbon diagram, like a Möbius strip, by taking the trace over a half-twist. Unfortunately, this would not typecheck, since $\tau_X \colon \overline{X} \to X$, and we cannot take the trace of this morphism. It seems that a *coherence* isomorphism $u_X \colon X \to \overline{X}$ is needed to identify the two sides invisibly. It can then be composed with τ, and we can take the trace of the composite. u must be invisible in the graphical calculus, very unlike τ.

Even without the ability of evaluating nonoriented diagrams, the graphical calculus of half-twists is useful. Let $\mathcal{C}$ be a half-ribbon category. Consider for example a ribbon graph, where vertices are labelled with morphisms in $\mathcal{C}$, and edges are labelled with objects. Choose an embedding in $\mathbb{R}^2 \times \mathbb{R}$ such that the projection on $\mathbb{R}^2$ is regular (i.e. no three ribbons project onto the same point and no singularities occur). We can then evaluate the ribbon graph in the graphical calculus of $\mathcal{C}$. It seems reasonable to conjecture that the evaluation will be invariant under isotopies. (Possibly, mild extra assumptions need to be made on $\mathcal{C}$.) The evaluation is then an invariant of the graph and the embedding.

This is interesting for the study of two-dimensional spin-sensitive state sum models [BT15]. Such a model can be defined by immersing a triangulated surface into $\mathbb{R}^3$

and labelling the ribbon graph corresponding to the triangulation with data from a special symmetric Frobenius algebra in $\mathrm{Vect}_{\mathbb{C}}$. Its evaluation takes the form of a state sum model, and is an invariant of the surface and its spin structure, which is defined as the pull-back of the canonical spin structure of $\mathbb{R}^3$, along the immersion. When a full-flegded graphical calculus of half-twists is available, we can consider Frobenius algebras internal to an arbitrary half-ribbon category and consider the invariant assigned to it. Such models are interesting from the viewpoint of mathematical physics, especially when studying defects on them, which also seems feasible in half-ribbon categories.

The author became interested in half-twists as a tool to study Noncommutative Geometry (NcG). Two-dimensional spin state sum models are, surprisingly, related to NcG ([FRS02, Section 6] and work by John Barrett, in preparation). The central objects of study are a $\star$-algebra (an involutive monoid in Vect), a Hilbert space with a real structure (an involutive object in Hilb) and certain operators acting on the Hilbert space. More strikingly, a graphical calculus has been developed recently for finite-dimensional NcG, and it involves half-twists. The author is confident that (finite dimensional) spectral triples internal to a half-ribbon category can be defined, such that original definition can be recovered as a special case. Nontrivial examples can then be introduced, such as Noncommutative Geometries with quantum symmetry groups.

What purposes do half-twists have for other areas of mathematics?

Braided monoidal categories are special cases of tricategories with one 1-morphism [CG07]. It seems possible that half-twists occur as higher-categorical structures as well. A comparison with the well-developed graphical calculus of Gray categories [BMS12] should shed light on this question.

The braid group acts on endomorphism sets in braided monoidal categories. It remains for future work to define a half-twist group or groupoid that generalises the braid group and acts on objects in a category with half-twist. Its relation to the braid group needs to be studied closely, since it could possibly lead to new insights on the field of braid group representations.

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